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ON THE YIELD CRITERION FOR SIMPLY REINFORCED CONCRETE SLABS IN PURE FLEXURE

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NOMENCLATURE

α = angle between the x-axis of the x-y-z coordinate system and the n-axis of the n-t-z coordinate system

β = second partial of the deflection function with respect to n

γ_{nt} = shearing strain in the t-direction on a cross-section with an exterior normal in the n-direction

γ_o = shearing strain at the neutral axis on a cross-section with an exterior normal in the n-direction

$\gamma_o(A)$ = a value of γ_o computed in application (A) of Appendix B

$\gamma_o(B)$ = a value of γ_o computed in application (B) of Appendix B

γ_{xy} = shearing strain in the y-direction on a cross-section with an exterior normal in the x-direction

ε_c = strain in the concrete under a uniaxial load

ε_i = principal strain (i = an integer)

ε_{ii} = uniaxial strain portion of normal strain on a cross-section with an exterior normal in the n-direction

ε_{pc} = yield strain in idealized concrete

ε_{ps} = yield strain in steel under uniaxial load

ε_{px} = yield strain in the x-direction reinforcing bar

ε_{py} = yield strain in the y-direction reinforcing bar

ε_s = strain in steel under uniaxial load

ε_{tc} = tensile rupture strain in idealized concrete

ε_{tt} = uniaxial strain portion of normal strain on a cross-section with an exterior normal in the t-direction

ε_{uc} = compressive rupture strain in idealized concrete

ε_x = axial strain in the x-direction reinforcing bar

ε_y = axial strain in the y-direction reinforcing bar

$\lambda, \lambda_1, \lambda_2$ = arbitrary positive numbers

μ = degree of orthotropy (ratio of M_{py} to M_{px})

ν = Poisson's ratio for concrete

ρ = second partial of the deflection function with respect to n and t

σ = normal stress

σ_c = stress in concrete under uniaxial load

σ_i, σ_{ii} = principal stress (i = an integer)

σ_{nn} = normal stress on a cross-section with an exterior normal in the n-direction

σ_p = plastic stress

σ_{pc} = plastic compressive stress of the idealized concrete

σ_{ps} = yield stress of steel

σ_{px} = yield stress of the y-direction reinforcing bar

σ_s = stress in the steel under uniaxial load

σ_{tc} = tensile rupture stress of the idealized concrete

σ_{tt} = normal stress on a cross-section with an exterior normal in the t-direction

σ_{xx} = normal stress on a cross-section with an exterior normal in the x-direction

σ_{yy} = normal stress on a cross-section with an exterior normal in the y-direction

τ_{nt} = shearing stress in the t-direction on a cross-section with an exterior normal in the n-direction

τ_p = plastic shearing stress

τ_{xy} = shearing stress in the y-direction on a cross-section with an exterior normal in the x-direction

φ = a function defined in equation (III.1.1) of Appendix B

ψ = second partial of the deflection function with respect to t

a = depth of equivalent rectangular stress block

a_i = linear function of the elements in matrix $[A]$ of equations (38) (i = an integer)

a_{11} = the first element in matrix $[A]$ of equations (38)

A_x = cross-sectional area of an x-direction bar

A_y = cross-sectional area of a y-direction bar

$[A]$ = a matrix defined in equations (38)

c = distance from the neutral axis to the point of ultimate compressive strain

C_{ep} = resultant compressive force of elastic-plastic model

C_{rp} = resultant compressive force of rigid-plastic model

d = distance from compressive surface of slab to the center-line of the reinforcing bar

e_{ii} = principal strain (i = an integer)

e_{nn} = normal strain on a cross-section with an exterior normal in the n-direction

e_{tt} = normal strain on a cross-section with an exterior normal in the t-direction

e_{xx} = normal strain on a cross-section with an exterior normal in the x-direction

e_{yy} = normal strain on a cross-section with an exterior normal in the y-direction

E_c = modulus of elasticity of the idealized concrete

E_{ij} = material constants (i and j are integers)

E_s = modulus of elasticity of steel

$[E]$ = the matrix of equations (32)

f'_c = uniaxial compressive strength of a 6" x 12" cylinder

f_{ct} = tensile strength of the idealized concrete

f_p = yield stress under uniaxial load

f_{px} = yield stress under uniaxial load of x-direction bar

f_{py} = yield stress under uniaxial load of y-direction bar

f_x = uniaxial stress in the x-direction bar

f_y = uniaxial stress in the y-direction bar

F_n = n-direction component of force per unit length of crack-line

F_t = t-direction component of force per unit length of crack-line

F_x = x-direction component of force per unit length of crack-line

F_{xn} = n-direction component of F_x

F_{xt} = t-direction component of F_x

F_y = y-direction component of force per unit length of crack-line

F_{yn} = n-direction component of F_y

F_{yt} = t-direction component of F_y

h = length of moment arm

k_{nt} = shearing strain at the neutral axis on a cross-section with an exterior normal in the n-direction

k_1 = an empirically determined constant that relates the depth of the rigid-plastic stress diagram to the distance between the neutral axis and the point of ultimate compressive strain

L = length of slab (in n-direction)

L_x = length of crack-line per x-direction bar

L_y = length of crack-line per y-direction bar

M = constant moment per unit width of cross-section applied along slab boundary

M_i = principal moment (i = an integer)

M_L = smallest moment in the imperfectly plastic region

M_{nn} = bending moment per unit length of crack-line

M_{nt} = twisting moment per unit length of crack-line

M_p = yield moment

M_{px} = yield moment per unit length of crack-line at $\alpha = 0°$

M_{py} = yield moment per unit length of crack-line at $\alpha = 90°$

M_U = largest moment in the imperfectly plastic region (ultimate moment)

M_{xx} = bending moment per unit width of cross-section having an exterior normal in the x-direction

M_{yy} = bending moment per unit width of cross-section having an exterior normal in the y-direction

M_{xy} = torsional moment per unit width of cross-section having an exterior normal in the x-direction

$M_{nn}^{(A)}$ = ultimate moment per unit width of cross-section along a crack-line (the superscript indicates that it is a value computed in application (A) of Appendix B)

$M_{nn}^{(C)}$ = ultimate moment per unit width of cross-section along a crack-line (the superscript indicates that it is a value computed in application (C) of Appendix B)

$M_{nn}^{(J)}$ = ultimate moment per unit width of cross-section along a crack-line (the superscript indicates that it is a value computed from a version of Johansen's criterion)

$M_{nn}^{(L)}$ = ultimate moment per unit width of cross-section along a crack-line (the superscript indicates that it is a value obtained in Lenkei's tests)

n, t, z = coordinates (designating position) of an orthogonal coordinate system in which the t-axis coincides with a crack-line

N = size of typical segment in the n-direction

q = applied load per unit area acting in the z-direction

Q_{nz} = z-direction shear force per unit width on a cross-section having an exterior normal in the n-direction

Q_{tz} = z-direction shear force per unit width on a cross-section having an exterior normal in the t-direction

S_x = spacing between x-direction bars

S_y = spacing between y-direction bars

T = size of typical segment in the t-direction

T_{xx} = x-direction component of force per x-direction bar

T_{xy} = y-direction component of force per x-direction bar

T_{yx} = x-direction component of force per y-direction bar

T_{yy} = y-direction component of force per y-direction bar

[T] = the matrix of equations (33) and (36)

w = deflection of the slab in the z-direction

W = slab width (in the t-direction)

x, y, z = coordinates (designating position) of an orthogonal coordinate system in which the directions of the x and y-axes coincide with the symmetry axes for an orthotropic material and with the direction of the reinforcing bars for reinforced concrete

Y = length of bar in a typical segment

z_p = distance from neutral axis to the point of compressive yield strain to the elastic-plastic model

SUMMARY

The yield criterion is a necessary element in the application of limit analysis to reinforced concrete slabs. Several yield criteria are available, but none of the criteria is able to predict all of the reported results obtained in experimental efforts to establish a yield criterion. The criterion of K. W. Johansen currently receives the widest practical use.

Among the experimental efforts are two different series of tests on reinforced concrete slabs subjected to uniaxial bending; each series contains slabs having orthotropic reinforcing. When Johansen's criterion is applied to the yield zone (yield-line) of the orthotropically reinforced slabs in each series and the computed bending moment is compared to the measured bending moment, the results of one series show good agreement between the computed and measured moments while the results of the other series show many measured moments with values significantly less than the computed values. The objective of this work is a rational explanation for the difference in behavior between these two groups of tests. Since the only essential difference in the two test set-ups is in the boundary conditions (in one set-up twisting is prevented), an explanation of these results requires a model that is able to describe the moment-curvature relationship for all values of moment up to and including the maximum moment and that is able to include the effect of the boundary conditions.

For the test set-up in which twisting is prevented, a solution analogous to one obtained by orthotropic plate theory is constructed for a reinforced concrete slab by introducing a modification of Kirchoff's assumptions which permits a uniform shearing strain in the uncracked portion of the concrete along a crack-line. The required moment-curvature model is completed by defining idealized constitutive relationships for the concrete and by defining a relationship between the strain on the cross-section and the force in a reinforcing bar. Finally, the maximum moment is defined by identifying it with the attainment of a failure state in the concrete, a state determined by using the maximum principal strain theory of failure. The resulting model provides a new criterion.

By using the new criterion it is possible to predict the difference in behavior which occurred in the two different series of tests. The lower values of maximum bending moment, observed when twisting is prevented, are attributed to the occurrence of a concrete tensile failure at the neutral axis prior to the occurrence of a concrete compressive failure at the slab's surface. The higher values of maximum bending moment, observed when twisting is permitted, are attributed to the redistribution of stress allowed by the twisting action until simultaneous failures, in tension at the neutral axis and in compression at the slab's surface, occur in the concrete.

CHAPTER I

INTRODUCTION

The approach currently being followed in establishing a yield criterion for reinforced concrete slabs relies on the idea that the behavior of a complex material like reinforced concrete can be adequately described in terms of the properties of a simpler, perhaps idealized, material and in terms of a body of theory already in existence. The areas of theory being utilized are the theory of thin plates and the theory of plasticity. It is assumed that the reader has some familiarity with the areas of theory involved, particularly of thin plate theory; nevertheless, an attempt will be made to keep to a minimum the necessary previous knowledge.

The Yield Criterion

Certain materials at ordinary temperatures exhibit the ability to undergo large deformations without rupture and with little increase in their load-carrying capacity. Such deformations are referred to as *plastic* deformations; and if the load remains constant during the deformation, the material is said to be *perfectly plastic* over that region of deformation. A demonstration of this phenomenon occurs in a simple tension test of a steel bar and is manifested in the stress-strain curve of the test. At a certain value of stress, called the *yield stress*, the curve suddenly becomes horizontal and maintains this state over a large range of strain values. A portion of a typical stress-strain curve, showing the perfectly plastic region, for a simple tension test of a steel bar is presented in Figure 1.

Since the transition from the pre-plastic region to the plastic region usually occurs quite rapidly, it can be characterized mathematically as a point of discontinuity in the derivative of the stress-strain relationship. Such a point of discontinuity can be located by specifying the state of stress at which it occurs; the collection of all such states of stress constitutes a yield criterion.

The yield criterion is usually expressed as a mathematical function relating the components of the state of stress. Any set of stress components that satisfies this relationship defines a point at which the plastic region begins. For the simple tension test the yield criterion can be expressed as

$$\sigma_p - \sigma = 0 \tag{1}$$

where σ is the normal stress on the cross-section of the bar and σ_p is the yield stress. It should be noted that in this particular case σ is a principal stress and that the other two principal stresses are zero. In general, the yield criterion would be expressed in terms of all three principal stresses. An example of a yield criterion expressed in terms of principal stresses σ_1, σ_2 and σ_3 is the widely used one of von Mises (1)*

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2(\sigma_p)^2 \, . \tag{2}$$

Von Mises' criterion reduces to the previously mentioned criterion

---

*Numbers in parentheses refer to references in "Literature Cited."

for the simple tension test when

$$\sigma_1 = \sigma \qquad (3)$$

$$\sigma_2 = 0$$

$$\sigma_3 = 0 \ .$$

The variables used to define a yield criterion need not have the dimensions of a stress; they may be what Prager (2) refers to as <u>generalized</u> stresses. Corresponding to a generalized stress there is a generalized strain. Moment per unit width of cross-section is the generalized stress in terms of which the yield criterion for plates and slabs is usually expressed; the corresponding generalized strain is the curvature.

The Flow Rule

A characteristic of the perfectly plastic region is the fact that the generalized stress is independent of the generalized strain, i.e., the generalized stress remains constant for all values of the generalized strain within the plastic region; consequently, specifying a particular state of stress which satisfies the yield criterion does not determine a unique state of strain. The flow rule is concerned with providing information about the state of strain that is associated with a particular state of stress at yield.

A widely used flow rule is the one used by von Mises (3) in his theory of plasticity. It states that the components of the strain vector in the plastic region are directly proportional to the partial derivatives of the yield function with respect to the stress components.

As an example of its application, consider von Mises' yield criterion as expressed in equation (2). The plastic strain components are given by

$$\begin{aligned} \epsilon_1 &= \lambda(2\sigma_1 - \sigma_2 - \sigma_3) \\ \epsilon_2 &= \lambda(2\sigma_2 - \sigma_3 - \sigma_1) \\ \epsilon_3 &= \lambda(2\sigma_3 - \sigma_1 - \sigma_2) \end{aligned} \tag{4}$$

where λ is an arbitrary positive factor of proportionality.

Koiter (4) has extended von Mises' concept so that a continuous function with discontinuities in the first derivative can be used to define a yield criterion. As an example consider a particular case of the maximum principal stress theory in which the yield criterion is expressed mathematically by

$$(\sigma_1 - \sigma_p)(\sigma_2 - \sigma_p)(\sigma_1 + \sigma_p)(\sigma_2 + \sigma_p) = 0 \tag{5}$$

and graphically (see Figure 2) by a square. Since there are only two variables in equation (5), the terms in parentheses, when equal to zero, may be thought of as pairs of simultaneous equations. Each pair of equations that can be satisfied simultaneously defines a corner of the square. For a state of stress at a corner, two of the terms in parentheses vanish. Using $\sigma_1 = \sigma_2 = \sigma_p$ as an example, Koiter would apply the flow rule separately to the first and second terms of equation (5) as follows

$$\varepsilon_1 = \lambda_1 \frac{\partial}{\partial\sigma_1} (\sigma_1 - \sigma_p) + \lambda_2 \frac{\partial}{\partial\sigma_1} (\sigma_2 - \sigma_p) \tag{6}$$

$$\varepsilon_2 = \lambda_1 \frac{\partial}{\partial\sigma_2} (\sigma_1 - \sigma_p) + \lambda_2 \frac{\partial}{\partial\sigma_2} (\sigma_2 - \sigma_p)$$

which, after differentiation, become

$$\varepsilon_1 = \lambda_1 \tag{7}$$

$$\varepsilon_2 = \lambda_2 .$$

Since λ_1 and λ_2 are arbitrary positive factors, the strain can have any direction between the normals to the lines formed in Figure 2 by the vanishing of the first two terms of equation (5). The permissible directions are illustrated in Figure 2 by the arrows.

Koiter's approach differs from von Mises' in that at yield a particular stress state can be associated with a family of strain directions rather than with a unique one and a particular strain direction can be associated with a family of stress states.

Limit Analysis

A structure is generally considered to have reached its useful limit when large deflections begin to accompany small increases in load; if perfectly plastic, the large deflections would occur without any increase in load. The load at which large deflections ensue is referred to as a limiting load, and the object of limit analysis is the determination of limiting loads.

In determining the limiting load of a structure composed of perfectly plastic materials, the upper-bound and lower-bound theorems are very important tools. These theorems can be stated as follows:

> <u>lower-bound theorem</u> - a load for which a stress field (including generalized stress fields) can be found that satisfies the equilibrium conditions and contains no points lying outside the surface specified by the yield criterion is a load that is equal to or less than the limiting load
>
> <u>upper-bound theorem</u> - a load determined by equating the internal and external work done by the application of an arbitrary plastic strain field which satisfies compatibility and kinematic boundary requirements is a load equal to or greater than the limiting load.

When proportional loading (loading determined by a single parameter) is postulated, the theorems can be used to determine upper and lower-bounds on the limiting load; and when these bounds coincide, the limiting load has been determined.

Yield-Line Theory

Yield-line theory was developed for and is primarily used on reinforced concrete slabs. It was introduced by A. Ingerslev (5) in 1921, and the major work in bringing it to its present state of development was done by K. W. Johansen (6).

Yield-line theory is a modification of limit analysis that uses the upper-bound approach. The plastic strains are assumed to be concentrated in very narrow bands, referred to as <u>yield lines</u>. Any configuration of yield-lines that permits the remaining parts of the slab to behave as a mechanism constitutes a plastic strain field satisfying compatibility and kinematic boundary requirements. Such a configuration of yield-lines is referred to as a collapse mechanism.

For each type of collapse mechanism, the mechanism giving the smallest upper-bound limiting load can be determined, thus defining the limiting load for a specific type of collapse mechanism; also, the smallest limiting load can be selected from those corresponding to different types of collapse mechanisms. It follows that the limiting load of a slab can be determined if the actual collapse mechanism is included among the types considered; that such is the case is a basic assumption of yield-line theory.

Significance of the Yield Criterion

Since the application of either the upper-bound or the lower-bound theorem requires a knowledge of the yield criterion, the results of their application is no better than the yield criterion used. If, then, one wishes to have confidence when applying limit analysis, and in particular when applying yield-line theory, a firmly established yield criterion is a necessity.

CHAPTER II

CURRENT STATE OF THE YIELD CRITERION

A mathematical analogy of a physical phenomenon attempts to describe the interrelationships that exist among the members of a chosen set of measurable variables involved in the phenomenon. Two basically different approaches by which to arrive at a mathematical description of a particular physical phenomenon are currently in general use. One approach is to collect, by actual measurement, many particular sets of the chosen variables and to use existing mathematical, or graphical, techniques to arrive at a mathematical form that will reproduce within acceptable limits the particular measured sets. In this approach there is no need to provide a physical interpretation of the mathematics, although one may exist. The problem solved is essentially a mathematical one - that of providing an approximate interpolation formula for a known set of data points. The other approach uses the physical event as a starting point. Observations of the event are utilized to deduce, or hypothesize, properties that the mathematical analogy should possess. Then, in conjunction with generally accepted mathematical formulations of related physical phenomena, a mathematical analogy is postulated. The validity of the proposed analogy is established by comparison with actual measurements.

Current formulations of the yield criterion for reinforced concrete slabs can be identified as belonging primarily to one or the other of these two approaches. Of these formulations only the one of

Baus and Tolaccia (7) can be classified as having used the first mentioned approach. Of the remaining formulations the criterion proposed by K. W. Johansen (6) has received the widest practical use; in fact, a number of the other formulations are equivalent formulations of this same criterion. For example, M. Save (8) has shown that the form of the criterion found in Nielsen's (9) work can be derived directly from Johansen's criterion; and in the closure of a discussion of their paper, Lenschow and Sozen (10) conceded that their proposed criterion was equivalent to Nielsen's and also to formulations presented by K. O. Kemp (11) and by A. Hillerborg (12). Consequently, of this group, only Johansen's criterion is discussed in detail; Nielsen's does differ with respect to the magnitude of the torsional moment, and his criterion is discussed to this extent. The only other criterion discussed specifically is the one proposed by M. W. Kwiecinski (13). It differs from the others in its treatment of the behavior of the reinforcing bars. Kwiecinski attempts to account for the reorientation of the reinforcing bar as it crosses an opening crack, whereas the others (with the exception of Baus and Tolaccia) assume that this phenomenon has an insignificant effect.

Criterion of K. W. Johansen

One of the earliest and simplest of the current criteria is the one proposed by K. W. Johansen for isotropic reinforcement. It is often presented, at least graphically, in a form analogous to the maximum normal stress theory mentioned in the Introduction, hence the term "square yield criterion" often used in referring to it. Using Koiter's approach and letting M_1 and M_2 be principal moments, Johansen's criterion becomes

$$(M_1 - M_p)(M_2 - M_p)(M_1 + M_p)(M_2 + M_p) = 0 \tag{8}$$

where M_p is the yield moment; however, one rarely sees it expressed in this form mathematically. The usual mathematical form is developed in what follows.

Before proceeding any further, a review of the rationale that permits a reinforced concrete slab to be treated as having continuously distributed properties along the width of a cross-section may be helpful. A reinforced concrete slab is composed of two materials, steel and concrete. The steel is distributed throughout the slab in the form of small bars placed in a grid pattern and is surrounded by the concrete. The concrete cracks at relatively low levels of tensile stress and, for most practical cases, usually cracks prior to yielding of the steel. Consequently, the critical cross-section is one that coincides with a crack, and the properties of this cracked section are utilized in reinforced concrete theory. If the forces in the bars crossing the crack are converted into an equivalent line-force along the width of the cross-section, then a reinforced concrete slab can be treated as having continuously distributed properties along the width of the particular cross-section under consideration.

Consider a cracked-section at some arbitrary angle to the direction of the reinforcement. Let the direction of the reinforcement coincide with an x-y coordinate system, and let the normal to the cracked section coincide with the n-axis and the crack-line coincide with the t-axis of an n-t coordinate system. Now let us define the following symbols:

α = angle between the x-axis and the n-axis

f_p = yield stress under uniaxial load

h = length of moment arm

A_x = cross-sectional area of an x-direction bar

A_y = cross-sectional area of a y-direction bar

F_n = n-direction component of force per unit length of crack-line

F_t = t-direction component of force per unit length of crack-line

F_x = x-direction component of force per unit length of crack-line

F_y = y-direction component of force per unit length of crack-line

F_{xn} = n-direction component of F_x

F_{xt} = t-direction component of F_x

F_{yn} = n-direction component of F_y

F_{yt} = t-direction component of F_y

L_x = length of crack-line per x-direction bar

L_y = length of crack-line per y-direction bar

M_{nn} = bending moment per unit length of crack-line

M_{nt} = twisting moment per unit length of crack-line

M_{px} = yield moment per unit length of crack-line at $\alpha = 0°$

M_{py} = yield moment per unit length of crack-line at $\alpha = 90°$

S_x = spacing between x-direction bars

S_y = spacing between y-direction bars

T_{xx} = x-direction component of force per x-direction bar

T_{xy} = y-direction component of force per x-direction bar

T_{yx} = x-direction component of force per y-direction bar

T_{yy} = y-direction component of force per y-direction bar.

Referring to Figure 3 the following geometrical relationships can be observed

$$L_x = S_x/\cos \alpha \qquad (9)$$

$$L_y = S_y/\sin \alpha$$

and utilizing the simplification of converting bar forces into an equivalent line-force gives

$$F_x = (T_{xx}/L_x) + (T_{yx}/L_y) \qquad (10)$$

$$F_y = (T_{yy}/L_y) + (T_{xy}/L_x) \ .$$

Assuming that a reinforcing bar crossing a crack-line behaves as if it were in a state of simple tension provides

$$T_{xy} = T_{yx} = 0 \qquad (11)$$

and assuming that a reinforcing bar crossing the crack in a yield line must be in the yielded state provides

$$T_{xx} = A_x f_p \qquad (12)$$

$$T_{yy} = A_y f_p \ .$$

Then, substitution of equations (9), (11) and (12) into equations (10) reduces them to

$$F_x = (A_x/S_x) f_p \cos \alpha \qquad (13)$$

$$F_y = (A_y/S_y) f_p \sin \alpha \ .$$

Referring now to Figure 4, the transformation of the equivalent line-force from x and y-direction components to n and t-direction components is given by

$$
\begin{aligned}
F_{xn} &= F_x \cos \alpha \\
F_{xt} &= -F_x \sin \alpha \\
F_{yn} &= F_y \sin \alpha \\
F_{yt} &= F_y \cos \alpha
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
F_n &= F_{yn} + F_{xn} \\
F_t &= F_{yt} + F_{xt} \; .
\end{aligned}
\tag{15}
$$

Substitution of equations (13) and (14) into equations (15) reduces them to

$$
\begin{aligned}
F_n &= (A_y/S_y)\, f_p \sin^2\alpha + (A_x/S_x)\, f_p \cos^2\alpha \\
F_t &= ((A_y/S_y)\, f_p - (A_x/S_x)\, f_p) \sin \alpha \cos \alpha \; .
\end{aligned}
\tag{16}
$$

Assuming that the moment arm is the same length for both, the bending and twisting moments per unit length of yield-line are given by

$$
\begin{aligned}
M_{nn} &= F_n h \\
M_{nt} &= F_t h
\end{aligned}
\tag{17}
$$

and substituting equations (16) into these gives

$$M_{nn} = (A_y/S_y)\ f_p h \sin^2\alpha + (A_x/S_x)\ f_p h \cos^2\alpha \ . \tag{18}$$

$$M_{nt} = ((A_y/S_y)\ f_p h - (A_x/S_x)\ f_p h) \sin\alpha \cos\alpha \ .$$

Assuming that the length of the moment arm is independent of the orientation of the yield-line to the reinforcement, the yield moments for yield-lines normal to the reinforcement directions are given by

$$M_{py} = (A_y/S_y)\ f_p h \tag{19}$$

$$M_{px} = (A_x/S_x)\ f_p h$$

and substitution of these into equations (18) gives

$$M_{nn} = M_{py} \sin^2\alpha + M_{px} \cos^2\alpha \tag{20}$$

$$M_{nt} = (M_{py} - M_{px}) \sin\alpha \cos\alpha$$

which is the general form of Johansen's criterion.

In order to go from the general form to the square yield form of the isotropic case, recall that

$$M_{py} = M_{px} = M_p \ . \tag{21}$$

Substituting this into the general form gives

$$M_{nn} = M_p \tag{22}$$

$$M_{nt} = 0$$

from which it can be seen that the moment along the yield-line is a principal one and that the orientation of the yield-line to the reinforcement is arbitrary. Remembering, however, that all reinforcement was assumed to be yielding in order to arrive at the general form expressed in equations (20), it is necessary to account for the physical fact that a yield-line can occur normal to one reinforcement direction without requiring the other reinforcement to yield. This condition occurs when $\alpha = 0^{\circ}$ and when $\alpha = 90^{\circ}$. Observation of equations (20) shows that M_{py} is arbitrary when $\alpha = 0^{\circ}$ and M_{px} is arbitrary when $\alpha = 90^{\circ}$; therefore not all reinforcement is required to yield in these two cases. This exception leads directly to the square yield form.

In the case of orthotropic reinforcement (different reinforcement in the x and y directions), M_{py} is not equal to M_{px}; and at yield, the cross-section can have both a bending and a twisting moment. Whether or not a twisting moment could exist along a yield line has been a matter of some controversy; however, recent tests by Lenschow and Sozen (14) and by Peter Lenkei (15) show that it can exist. The possibility of a principal curvature on other than a principal moment cross-section has been demonstrated in orthotropic plate theory; therefore the acceptance in yield-line theory of a principal curvature along a yield-line is compatible with both Johansen's criterion and thin plate theory.

Johansen conducted no comprehensive test program to verify his yield criterion explicitly. He did conduct tests to determine yield principal moments on cross-sections perpendicular to the reinforcement while holding the minor principal moment equal to zero, i.e. tests on wide, shallow beams. These tests demonstrated that such reinforced concrete members did have properties which closely approximated the perfectly plastic condition. He also showed that the application of yield-line theory, using his criterion, was able to predict reasonably well the load carrying capacities of slabs tested by the German Reinforced Concrete Board and by himself (6), which may be interpreted as an implicit verification of the yield criterion.

Another widely used form of the yield criterion, which was offered by Nielsen (9), among others, and which M. Save (8) has shown to be equivalent to Johansen's, as expressed in terms of the moments on cross-sections whose normals are parallel to the directions of the reinforcement, i.e. the x and y axes. This form of the yield criterion is expressed as follows

$$(M_{px} - M_{xx})(M_{py} - M_{yy}) - (M_{xy})^2 = 0 \; . \tag{23}$$

Equation (23) implies a positive bending moment on the yielding cross-section; a similar equation for a negative moment at yield is

$$(M'_{px} + M_{xx})(M'_{py} + M_{yy}) - (M_{xy})^2 = 0 \tag{24}$$

where the prime indicates negative moment magnitudes. In this form the generalized flow law of the plastic potential can be readily applied, and solutions in the theory of plasticity become available for use on reinforced concrete slabs.

Criterion of M. P. Nielsen

Although the form of Nielsen's criterion is equivalent to Johansen's, it differs with respect to the value of the torsional moment on the cross-sections normal to the reinforcing bars. Nielsen's criterion reduces to Johansen's as a special case, but in general it predicts smaller values of torsional moment. The lower values of

Nielsen are not unexpected since his criterion was derived using a lower-bound approach.

Nielsen's criterion rests on the idealization of steel and concrete into rigid-plastic materials. Using these idealizations he constructs a statically admissible stress field that nowhere exceeds the plastic limits.

Since Nielsen's criterion differs from Johansen's with respect to the torsional moment and since Nielsen's tests were designed to provide information about the magnitude of pure torsional moment on cross-sections normal to the reinforcing bars, only this particular case of his criterion is presented. The stress field consists of two disconnected plane stress fields in the concrete, one at the top and one at the bottom of the slab, and uniaxial stress fields in the reinforcing bars. The two principal stresses in both concrete plane stress fields are respectively equal to zero and to the compressive plastic stress; the uniaxial stresses in the reinforcing bars are equal to the tensile plastic stress. The depths of the two concrete stress fields are assumed to be equal, and the orientation of the concrete principal stresses to the direction of the reinforcing bars is such that the compressive force in the concrete on the cross-section normal to the reinforcing bar is equal to the tensile force in the bar. Consequently, the bending moment on this cross-section is zero, but a torsional moment is present due to the shearing stresses in the concrete. According to this stress field the torsional moment can be expressed, for all practical purposes, as

$$M_{xy} = \sqrt{\mu}\ M_{px} \tag{25}$$

where

$$\mu = M_{py}/M_{px} \tag{26}$$

and M_{py} and M_{px} are the yield principal moments on cross-sections normal to y and x-direction reinforcement respectively.

Nielsen (16) conducted experiments whose results were in excellent agreement with his criterion.

Criterion of M. W. Kwiecinski

Kwiecinski justifies the difference between his criterion and others on the basis of the two following assumptions:

(1) the line of action of the force in a reinforcing bar crossing a crack deviates at yield from the original direction of the bar

(2) the change in direction of the forces in the bars crossing a crack will adjust so that a principal moment is maintained along the yield line.

Johansen assumes that the line of action of the force in the bar does not change. Wood (17) discusses a "kinking" theory that assumes the new line of action of the force to be perpendicular to the crack. Kwiecinski's assumptions produce a criterion that lies somewhere between these two; consequently, it is referred to as a "partial-kinking" criterion.

The partial-kinking criterion for isotropic reinforcement has the following form

$$M_{nn} = (\cos \alpha \sqrt{1 - A^2 \sin^2 \alpha} + \sin \alpha \sqrt{1 - A^2 \cos^2 \alpha})M_p \qquad (27)$$

where A is an empirically determined constant that may vary between zero and one. When A is one, the criterion reduces to Johansen's criterion; when A is zero, it reduces to the kinking criterion described by Wood.

Kwiecinski (18) conducted tests on simple-spanning slabs with reinforcement at angles of 0, 15, 30 and 45 degrees to the span direction in order to verify his theory, and according to his interpretation, the results supported his theory. Since the number of bars crossing a yield-line was small and since bond failures occurred along the slab edges, it was necessary for him to define an effective length of yield-line in order to get his results. Furthermore, there was no proof that strain hardening did not occur. In view of this, some question exists with respect to the degree of verification provided by his tests. Some supporting evidence with respect to the 45-degree test results is offered by Wood (17), who comments on the fact that a 16 percent increase in yield moment was observed in a number of tests conducted at the Building Research Station in England; Kwiecinski's results showed an 18.8 percent increase.

Criterion of Baus and Tolaccia

Square slabs subjected to uniformly distributed moments on opposite edges were used to produce the data from which their criterion was derived. The measured variables were the applied moments on the edges and the curvature at an interior point on the slab. The yield moment was identified by utilizing the measured moment-curvature relationship to determine the moment at which large increases in curvature ensued.

The measured moments were principal ones, hence their criterion is expressed in terms of the two principal moments which they have designated as M_1 and M_2. Since only the form of their criterion is of importance in what follows, no attempt will be made to present their criterion per se. The criterion consists of eight separate linear relationships in M_1 and M_2; its form is depicted graphically in Figure 5. Each of the eight linear relationships is a function of the angle between the direction of the reinforcement and the direction of the principal moment. Consequently, the slopes of the linear relationships are, in general, different for each angle, but they will not change signs.

The critical hypothesis underlying their approach is the assumption that the behavior of the slab element in their tests is representative of the behavior of a general element in the interior of a slab. This assumption is placed in doubt by the fact that the test element is banded by a region of heavier reinforcement and by the fact that the method of applying the load through clamped-on cantilevered arms along the edges greatly reduces the number of possible paths that the yield-lines may take out to the boundaries.

Many of Baus and Tolaccia's test values are significantly greater than the values predicted by Johansen's criterion (Johansen's values are shown as a dashed line in Figure 5). They attribute the larger values observed in their tests to a phenomenon referred to as "perpendicular plastification." Its effect is to change the line of action of the force in the reinforcing bar from one along the axis of the bar to one perpendicular to the opening crack which the bar is crossing. This phenomenon is also called "kinking" and was referred to previously in discussing the criterion of M. W. Kwiecinski.

Comparisons of Some Reported Experimental Results

A number of different investigators (7),(15),(16),(18),(19) conducted experiments designed to provide information about the yield criterion. There is considerable variation in their reported results, and unfortunately it occurs even in what appear to be equivalent tests.

In what follows, several phrases are used whose meaning may be unfamiliar or unclear to the reader; these phrases and their meanings are:

simply reinforced slab - all of the reinforcement is on one side of the middle surface of the slab

doubly reinforced slab - some reinforcement is on both sides of the middle surface of the slab

degree of orthotropy - the ratio of the yield moments associated with the cross sections that are perpendicular to the reinforcement directions. The ratio is usually expressed so that it is equal to or less than one.

Baus and Tolaccia as well as Lenschow and Sozen conducted tests in which equal biaxial moments of like signs were applied to simply reinforced test specimens. Using the values predicted by Johansen's criterion as a basis for comparing the test results, the two values reported by Baus and Tolaccia exceeded Johansen's predicted values by 28 and 29 percent, while Lenschow and Sozen reported six values that exceeded Johansen's predicted values by percentages ranging from only

1 to 5 percent. Lenschow and Sozen (19) suggest that a sizeable portion of the larger values reported by Baus and Tolaccia can be attributed to the band of heavier reinforcement that surrounds their test specimen, yet Nielsen's test specimens are surrounded by steel channels and no such large values appear in his reported results. Baus and Tolaccia attribute their higher results to kinking of the reinforcing bars, but no such kinking effect manifested itself in Lenschow and Sozen's work.

Baus and Tolaccia, Lenschow and Sozen and also M. P. Neilsen conducted tests in which equal moments of opposite sign were applied to doubly reinforced test specimens. Baus and Tolaccia used the same reinforcement in all their tests but changed the angle between the principal moments and the direction of the reinforcing bars; Nielsen used the same angle between reinforcing bars and principal moments in all his tests but changed the reinforcement; Lenschow and Sozen changed both in their tests. Baus and Tolaccia's values exceeded the values of Johansen's criterion by amounts ranging from 5 to 25 percent; Lenschow and Sozen's values ranged from 3 percent below to 5 percent above, and Nielsen's values ranged from 7 percent above to 21 percent below. For the case of principal moments at 45 degrees to the reinforcing bars, there are no tests in which all investigators had a reinforcement pattern with the same degree of orthotropy; however, Lenschow and Sozen with a degree of orthotropy of 1.0 and Nielsen with a degree of orthotropy of 0.74 reported results that exceeded Johansen's predicted values from 1 to 5 percent, while Baus and Tolaccia with an intermediate degree of orthotropy of 0.82 reported values that exceeded Johansen's by 16 percent. For a degree of orthotropy of 0.25, Lenschow and Sozen's values were

3 percent greater than those of Johansen's criterion; Nielsen's were 21 percent below. Nielsen's criterion predicts the low values that occurred in his tests, but it does not explain why similar low values did not occur in Lenschow and Sozen's tests. With regard to Baus and Tolaccia's higher values, nothing more can be added to the comments already made in the previous paragraph. It is worth noting, however, that both Baus and Tolaccia's high values and Nielsen's low values occurred on test specimens that were surrounded with a band of heavier reinforcement.

Baus and Tolaccia, Lenschow and Sozen, M. W. Kwiecinski and P. Lenkei conducted tests in which uniaxial moments were applied to simply reinforced test specimens. All of the investigators changed the angles between the applied moment and the reinforcing bars. Lenschow and Sozen and P. Lenkei also changed the degree of orthotropy in their reinforcement, but Baus and Tolaccia and M. W. Kwiecinski did not. Both Lenschow and Sozen's results and Baus and Tolaccia's results were within a variation of 5 percent above and 5 percent below the values of Johansen's criterion. Kwiecinski's results were equal to or greater than Johansen's values with the largest value being 18.8 percent above. Lenkei's values varied from 5 percent above to 16 percent below (excluding some tests on specimens using lightweight aggregate whose values went to 21 percent below).

Lenkei's lowest values occurred during tests on slabs having a degree of orthotropy of 0.45 and with an angle of 45 degrees between the yield-line and each reinforcing bar direction, yet Lenschow and Sozen reported a value (also their lowest value) only 3 percent below

Johansen's criterion for a slab having a degree of orthotropy of 0.5 and with angles of 40 and 50 degrees between the yield-line and the respective reinforcing bar directions. Lenkei attributes his lower values to the existence of a twisting moment in the yield-line, but Lenschow and Sozen's slabs also had twisting moments in the yield-line and no such low values occurred.

Kwiecinski's largest value occurred on a slab with an angle of 45 degrees between the yield-line and each reinforcing bar direction; the slab was isotropically reinforced. He attributes the higher value to the effect of kinking in the reinforcement, but Lenschow and Sozen also tested a slab with isotropic reinforcement and steel at 45 degrees to the yield-line and no such increase occurred. It is also interesting to note that Baus and Tolaccia, who attributed increases in some of their other tests to the "kinking" phenomenon, observed no increase in their tests of this type.

Summary

Some experimental results support each of the current yield criteria, but none of the current yield criteria is able to predict all of the reported experimental results.

The reported experimental results contain several paradoxes.

The criterion of K. W. Johansen currently receives the widest practical use, and until the experimental results reported by Nielsen and those reported by Lenkei, it seemed to be a conservative criterion, i.e., the absolute value of the predicted result being equal to or less than the absolute value of the experimental result.

CHAPTER III

OBJECTIVE AND METHOD OF ATTACK

Statement of Objective

In formulating a yield criterion for reinforced concrete slabs, it is generally taken as axiomatic that associated with each orientation of the yield-line to the reinforcement there is only one value of plastic bending moment per unit length of yield-line acting in the positive direction (and also only one value in the negative direction) and that the magnitude of this moment is dependent only upon the material properties and their arrangement in the cross-section. When one compares the experimental results reported in the previous chapter, several examples appear which suggest the possibility that this axiom may not be valid in all cases. The best example occurs when the results of Lenschow and Sozen's tests in uniaxial bending are compared with the results of Lenkei's tests.

In each test set-up the slab was supported along four parallel lines and a uniaxial bending moment was applied to the central portion of the slab through these supports. In Lenschow and Sozen's set-up all four supports were free to rotate about an axis perpendicular to the support lines, while in Lenkei's the two interior supports were prevented from rotating about this axis. Consequently, Lenschow and Sozen's set-up forced the principal moment to be parallel to the support line while permitting the yield-line to form at any angle; Lenkei's set-up forced the yield-line to form parallel to the support lines while providing a reactive support system for any torsional moments that might develop in the yield-line. In each case the experimental information sought was the magnitude of the moment along a yield line; this information can be determined by using either set-up. The slabs were simply reinforced and the range of orthotropy investigated was about the same. In both set-ups slabs with isotropic reinforcement were tested; Lenschow and Sozen tested slabs with a degree of orthotropy of 0.5 while Lenkei tested slabs with degrees of orthotropy of 0.45 and 0.6. Even the percentages of reinforcement did not vary greatly; Lenschow and Sozen's percentages varied between 1.0 and 0.4 percent, whereas Lenkei's varied between 0.9 and 0.2 percent.

Considering the similarities in the properties of the test specimens in conjunction with the position that the moment in a yield-line has a unique value which is independent of the boundary conditions, the test results should display similar characteristics. Yet when the yield-line moment values computed by Johansen's criterion are compared to the measured yield-line moment values within each group of tests, Lenkei's results display an entirely different trend than Lenschow and Sozen's results. Lenkei's results show that the measured moment becomes smaller as the degree of orthotropy becomes smaller (i.e. moves further from the isotropic case), the lowest values being 87 percent of Johansen's for a degree of orthotropy of 0.6 and 84 percent of Johansen's for a degree of orthotropy of 0.45. In tests having a degree of orthotropy of 0.5, Lenschow and Sozen's results show no such comparable decrease, their lowest value being 97 percent of Johansen's.

It would appear then, at least on the basis of these results, that the moment in a yield line is dependent not only upon the material properties and their arrangement in the cross-section but also upon the boundary conditions. It is the objective of this dissertation to offer a rational explanation for the difference in behavior between these two groups of tests.

Method of Attack

Since the difference between the two test set-ups was in the boundary conditions and since the measured yield moment was apparently affected by this difference, an explanation of these results will require a model that is able to describe the moment-curvature relationship for all values up to and including the plastic moment and that is able to include the effect of the boundary conditions.

In the Introduction a yield criterion was described as a mathematical function that defines the onset of the plastic region. It is assumed that the onset occurs abruptly enough to be identified with a particular state of generalized stress. The nature of the pre-plastic region is dependent upon the material being described. For at least a portion of the pre-plastic region, the stress-strain relationships of both concrete and steel can be adequately represented by linear functions. As long as the functions are linear, orthotropic plate theory provides a possible model for that portion of the pre-plastic region; in fact, M. T. Huber* has already applied this theory to reinforced concrete slabs. Consequently, the application of orthotropic plate theory to the particular case of Lenkei's test set-up serves as the

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*See S. Timoshenko and S. Woinowsky-Krieger (20), p. 366.

starting point for this study. The next step will be dependent upon the results of this first step, and each succeeding step will be dependent upon the previous step.

CHAPTER IV

ORTHOTROPIC PLATE THEORY AND REINFORCED CONCRETE SLABS

An orthotropic material is an anisotropic material having three planes of symmetry with respect to its elastic properties. Letting these planes coincide with the coordinate axes of an x-y coordinate system, the relations between the stress and the strain components have the following form

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{12} & E_{22} & 0 \\ 0 & 0 & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ \gamma_{xy} \end{bmatrix} \qquad (28)$$

The strain at any point on a slab cross-section having an exterior normal in a coordinate direction is related to the slab's behavior by Kirchoff's assumptions

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ \gamma_{xy} \end{bmatrix} = z \begin{bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x \partial y \end{bmatrix} \qquad (29)$$

where w is a function of x and y, expressed notationally as w(x,y), that describes the deflection of the slab in the z-direction at any point (x,y). Substituting Kirchoff's relations into equations (28) gives

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} z\,E_{11} & z\,E_{12} & 0 \\ z\,E_{12} & z\,E_{22} & 0 \\ 0 & 0 & 2z\,E_{33} \end{bmatrix} \begin{bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ \partial^2 w/\partial x \partial y \end{bmatrix} \tag{30}$$

The expressions for bending and torsional moments on a cross-section of unit width are

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \begin{bmatrix} \int z\,\sigma_{xx}\,dz \\ \int z\,\sigma_{yy}\,dz \\ \int z\,\tau_{xy}\,dz \end{bmatrix} . \tag{31}$$

Substitution of equations (30) into equations (31) permits the expression of the bending and torsional moments in terms of the slab's thickness, of the material's properties and of the curvatures and twists of the slab's neutral surface, namely

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \begin{bmatrix} E_{11}\int z^2 dz & E_{12}\int z^2 dz & 0 \\ E_{12}\int z^2 dz & E_{22}\int z^2 dz & 0 \\ 0 & 0 & 2\,E_{33}\int z^2 dz \end{bmatrix} \begin{bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ \partial^2 w/\partial x \partial y \end{bmatrix} . \tag{32}$$

Since the material planes of symmetry and the coordinate axes of a particular problem statement need not coincide, convert the moment expressions in the x-y system to their equivalents in an n-t system at some angle α to the x-y system. The moment transformation is

$$\begin{bmatrix} M_{nn} \\ M_{tt} \\ M_{nt} \end{bmatrix} = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & -\sin 2\alpha \\ \sin^2\alpha & \cos^2\alpha & \sin 2\alpha \\ \frac{\sin 2\alpha}{2} & -\frac{\sin 2\alpha}{2} & \cos 2\alpha \end{bmatrix} \begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} \tag{33}$$

and the coordinate transformation is

$$\begin{bmatrix} n \\ t \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} . \tag{34}$$

The partial derivatives are transformed by repeated application of the "chain rule" for differentiation as follows

$$\begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial n}{\partial x}\frac{\partial n}{\partial x} & \frac{\partial t}{\partial x}\frac{\partial t}{\partial x} & 2\frac{\partial n}{\partial x}\frac{\partial t}{\partial x} \\ \frac{\partial n}{\partial y}\frac{\partial n}{\partial y} & \frac{\partial t}{\partial y}\frac{\partial t}{\partial y} & 2\frac{\partial n}{\partial y}\frac{\partial t}{\partial y} \\ \frac{\partial n}{\partial x}\frac{\partial n}{\partial y} & \frac{\partial t}{\partial x}\frac{\partial t}{\partial y} & \frac{\partial n}{\partial x}\frac{\partial t}{\partial y} + \frac{\partial n}{\partial y}\frac{\partial t}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 w}{\partial n^2} \\ \frac{\partial^2 w}{\partial t^2} \\ \frac{\partial^2 w}{\partial n \partial t} \end{bmatrix} \tag{35}$$

Performing, on equations (34), the partial differentiation operations indicated in each element of the matrix in equations (35) and replacing each element with the results of the operations gives

$$\begin{bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ \partial^2 w/\partial x \partial y \end{bmatrix} = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & -\sin 2\alpha \\ \sin^2\alpha & \cos^2\alpha & \sin 2\alpha \\ \frac{\sin 2\alpha}{2} & \frac{-\sin 2\alpha}{2} & \cos 2\alpha \end{bmatrix} \begin{bmatrix} \partial^2 w/\partial n^2 \\ \partial^2 w/\partial t^2 \\ \partial^2 w/\partial n \partial t \end{bmatrix} \tag{36}$$

which is the same transformation matrix as the one required for moments.

Now, substitution of equations (36) into equations (32) and then substitution of these into equations (33) permits the expression of the bending and torsional moments in an n-t coordinate system in terms of the curvatures, the twists and the angle between the n-t axes and the symmetry axes of the material. In order to express this in a more compact form, let the matrix of equations (32) be represented by the symbol [E] and the matrix of equations (33) and (36) by [T]. Using this notation the expression resulting from the substitutions just described is

$$\begin{bmatrix} M_{nn} \\ M_{tt} \\ M_{nt} \end{bmatrix} = [T][E][T] \begin{bmatrix} \partial^2 w/\partial n^2 \\ \partial^2 w/\partial t^2 \\ \partial^2 w/\partial n \partial t \end{bmatrix} \tag{37}$$

or, in still more compact form

$$\begin{bmatrix} M_{nn} \\ M_{tt} \\ M_{nt} \end{bmatrix} = [A] \begin{bmatrix} \partial^2 w/\partial n^2 \\ \partial^2 w/\partial t^2 \\ \partial^2 w/\partial n \partial t \end{bmatrix} \tag{38}$$

where $[A] = [T][E][T]$.

The differential equations of equilibrium must also be satisfied; these are

$$\frac{\partial Q_{nz}}{\partial n} + \frac{\partial Q_{tz}}{\partial t} + q(n,t) = 0$$

$$Q_{nz} = \partial M_{nn}/\partial n - \partial M_{nt}/\partial t \tag{39}$$

$$Q_{tz} = \partial M_{tt}/\partial t - \partial M_{nt}/\partial n$$

where $q(n,t)$ = applied load per unit area acting in the z-direction

Q_{nz} = z-direction shear force per unit width on a cross-section having an exterior normal in the n-direction

Q_{tz} = z-direction shear force per unit width on a cross-section having an exterior normal in the t-direction

The first of equations (39) is usually replaced by an equivalent equation obtained by differentiation and substitution of the last two equations into the first, which results in the well-known expression

$$\frac{\partial^2 M_{nn}}{\partial n^2} + \frac{\partial^2 M_{tt}}{\partial t^2} - 2\frac{\partial^2 M_{nt}}{\partial n \partial t} = -q(n,t) \ . \tag{40}$$

It is also implied in this development that M_{tn} is equal to $-M_{nt}$ and that the resultant force per unit width of cross-section is equal to zero in the n-direction and in the t-direction.

Application of Orthotropic Plate Theory to Lenkei's Test Set-Up

Lenkei's test set-up is equivalent to the case of a rectangular slab having along two opposite edges a continuous simple support offering no resistance to translation parallel to the n-t plane, having no support along the other two edges and having a uniform bending moment applied along the supported edges. A plan view of this case, showing the x-y and the n-t coordinate systems and the resultants of the uniform bending moment, is depicted in Figure 6.

The boundary conditions along the simply supported edges consist of a constant bending moment per unit length of edge and zero deflection of the edge in the z-direction, expressed notationally as follows

$$M_{nn}(0,t) = M_{nn}(L,t) = M \tag{41}$$

$$w(0,t) = w(L,t) = 0$$

where M is the value of the uniformly applied moment. Following classical thin plate theory,* the boundary conditions along the free edges are

$$M_{tt}(n,0) = M_{tt}(n,W) = 0 \tag{42}$$

$$Q_{tz}(n,0) - \frac{\partial M_{nt}}{\partial t}(n,0) = 0$$

$$Q_{tz}(n,W) - \frac{\partial M_{nt}}{\partial t}(n,W) = 0 .$$

For this case of loading, equation (40) becomes

$$\frac{\partial^2 M_{nn}}{\partial^2 n} + \frac{\partial^2 M_{tt}}{\partial^2 t} - 2\,\frac{\partial^2 M_{nt}}{\partial n \partial t} = 0 . \tag{43}$$

After performing, on equations (38), the differentiations indicated in equation (43) and substituting the results back into equation (43), the following form results

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*See S. Timoshenko and S. Woinowsky-Krieger (20), pp. 83-86.

$$a_1 \frac{\partial^4 w}{\partial n^4} + a_2 \frac{\partial^4 w}{\partial n^3 \partial t} + a_3 \frac{\partial^4 w}{\partial n^2 \partial t^2} + a_4 \frac{\partial^4 w}{\partial n \partial t^3} + a_5 \frac{\partial^4 w}{\partial t^4} = 0 \qquad (44)$$

where the a_i's are linear functions of the elements in matrix [A] of equations (38).

Now consider the following expression for the deflection

$$w(n,t) = \frac{M}{2a_{11}} (n^2 - nL) \qquad (45)$$

where a_{11} is the first element in matrix [A]. This expression will satisfy the equilibrium condition - equation (44) - and the boundary conditions - equations (41) and (42) - with the exception of the first row of equations (42). If the slab is sufficiently wide, the strains will approach those determined by using equation (45);* thus the fact that the first row of equations (42) is not satisfied can be circumvented. That equation (45) represents an adequate solution is the viewpoint adopted herein.

Inconsistency Between the Orthotropic Plate Solution and the Behavior of a Simply Reinforced Concrete Slab in the Neighborhood of a Crack-Line

In order to simplify the presentation, only a single layer of tensile reinforcement will be discussed initially; the effect of the additional layer of tensile reinforcement will be considered later.

Consider the particular case of the previous section when the slab is composed of reinforced concrete and is in the cracked state. Let the orientation, spacing and size of the bar in the single layer

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*See S. Timoshenko and S. Woinowsky-Krieger (20), p. 4.

of reinforcement be the same throughout the slab. A plan view of the slab giving reference coordinate systems and showing, in dashed lines, the pattern of the reinforcement is depicted in Figure 7. Also shown in Figure 7, depicted as a shaded area, is a typical segment of slab which will be extracted for more detailed consideration. The segment is referred to as typical because it is the smallest repeating subdivision of a cross-section taken parallel to the supported edges, i.e. parallel to the t-axis. Since the solution, equation (45), from orthotropic plate theory is independent of t, each of these typical segments will be subjected to the same conditions; therefore what holds for one such segment holds for all and only one needs to be considered.

Remembering that the critical cross-section is one that coincides with a crack, the typical cross-section is chosen so that it contains a crack-line. It is assumed that the crack-line will occur parallel to the supported edges, which is reasonable since such lines are also lines of principal curvature in the orthotropic plate solution; furthermore, the physical validity of the assumption can be demonstrated by actual test.

In applying orthotropic plate theory to a reinforced concrete slab, reinforced concrete is considered to have an equivalent representation in terms of orthotropic material. One concept of such an equivalent representation is a combination of two orthotropic materials, one for the concrete and one for the steel, having coinciding directions of symmetry axes and having the same flexural properties on cross-sections normal to the symmetry axes as reinforced concrete has on cross-sections normal to the reinforcement directions. Figures 8a and 8b show two different enlarged perspective views of the typical segment shown as a shaded area in the plan view of Figure 7; Figure 8a depicts a segment composed of the two orthotropic materials and Figure 8b depicts a segment composed of the reinforced concrete. Both segments are depicted in accordance with the principal strains indicated by the orthotropic plate solution, i.e.

$$\begin{bmatrix} e_{nn} \\ e_{tt} \\ \gamma_{nt} \end{bmatrix} = z \begin{bmatrix} M/a_{11} \\ 0 \\ 0 \end{bmatrix} . \tag{46}$$

Since the critical cross-section lies in the crack-line, no forces are shown in Figures 8a and 8b.

Figures 8c and 8d show the resultant force systems that occur on the respective cross-sections through the crack as a result of the principal strains prescribed in equations (46). According to orthotropic plate theory, the two orthotropic materials in Figure 8c have force components normal to and parallel to the cross-section which form a pair of force couples - the bending and twisting moments on the cross-section. In Figure 8d, the steel bar of the reinforced concrete segment has a resultant force system similar to its corresponding orthotropic material (force components normal to and parallel to the cross-section) but the concrete, being an isotropic material in a state of principal strain, has only a force component normal to the cross-section. Since the supports provide no reactive forces parallel to the n-t plane, the reinforced concrete slab is not in equilibrium when the strains indicated by the orthotropic plate solution exist in the slab.

It has been suggested by Wood (17) that the force in the bar could become normal to the cross-section through "kinking," since it must kink to some degree in order to satisfy compatibility requirements as the crack opens. If Wood's suggestion is correct, the reinforced concrete slab could be in equilibrium when the strains indicated by the orthotropic plate solution exist in the slab, since the bar force component parallel to the cross-section would be zero. Wood's suggestion was tested experimentally by Dempsey and Howell* at the Georgia Institute of Technology. They ran simple tension tests up to rupture on reinforcing bars embedded in a pre-cracked block of concrete. The tension force was applied normal to the crack, the reinforcing bars were embedded at various angles to the crack and the two halves of the concrete block were prevented from translating relative to one another along the crack as the tension force was applied. By measuring the applied tension force and the forces required to prevent translation, the bar force components perpendicular and parallel to the crack could be computed from the equations of equilibrium. Only when the bar was normal to the crack did the force component parallel to the crack have a value of zero prior to rupture of the bar.

It follows, then, that the strain conditions indicated by the orthotropic plate solution lead to a violation of equilibrium in the reinforced concrete slab except for the case when the reinforcing bar

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*W. J. Dempsey and H. W. Howell are graduate students at Georgia Institute of Technology. The tests were conducted in partial fulfillment of the requirements for the degree of Master of Science in Civil Engineering.

is normal to the crack.

If another layer of tensile reinforcement is added at right angles to the original layer, the force components normal to and parallel to the crack result from the algebraic sum of the components in each layer. Since in each layer the components parallel to the crack are oppositely directed, the magnitude of the unbalanced force parallel to the crack is reduced by adding another layer, and for certain distributions of reinforcement the unbalance will be reduced to zero. With the exception, then, of those distributions of reinforcement in which a zero value for the force component parallel to the crack occurs, the conclusions reached for a single layer are valid for two layers at right angles.

A Modification of Kirchoff's Assumptions

If the solution of orthotropic plate theory is to be applicable to the reinforced concrete slab without violating equilibrium, a force component parallel to the crack must occur in the concrete without the occurrence of any twisting, i.e. the value of $\partial^2 w/\partial n \partial t$ must be zero. Physically, this means that the concrete on one side of the crack must undergo a translation along the t-axis with respect to the concrete on the other side of the crack (see Figure 9). In terms of plate theory this results in a modification of Kirchoff's assumptions - the addition of a term in the expression for shearing strain as follows

$$\gamma_{nt} = 2\ z\ (\partial^2 w/\partial n \partial t) + k_{nt}\ . \tag{47}$$

This modification permits the shearing strain in the concrete to have a non-zero value while $\partial^2 w/\partial n \partial t$ has a zero value.

Extension to More General Considerations

Consider Lenkei's test set-up if, instead of lying in a plane parallel to the n-t plane, the supports are assumed to have an initial rotation relative to one another about an axis parallel to the n-axis. The twist ($\partial^2 w/\partial n \partial t$) must now have a non-zero value in order to satisfy the boundary conditions. Several things about the yield state along a crack-line can be inferred from this consideration and from the modification of Kirchoff's assumptions.

That a yield-line is a line of principal curvature is a commonly accepted view. The introduction of an initial twist into the supports of Lenkei's test set-up suggests that this is not necessarily true. The kinematics of the test set-up forces the yield-line to form parallel to the supports, but the initial twist prevents the curvature from being a principal one.

The possibility of twisting along a yield-line suggests that more than one magnitude of yield moment can be associated with a particular yield-line. Having only one magnitude of yield moment associated with a particular yield-line would require that the yield moment be independent of the amount of initial twist in the supports, which seems rather unlikely.

The existence of a shearing strain at the neutral axis, resulting from the modified Kirchoff's assumptions, suggests that for some cases the tensile, not the compressive, strength of the concrete controls the maximum attainable moment along a crack-line.

An investigation into these inferences as possible explanations for the difference in behavior between the tests of Lenkei and those of Lenschow and Sozen is the next logical step. At this point it has been shown that the strains and force systems in a reinforced concrete slab are analogous to those in an orthotropic slab if a modified form of Kirchoff's assumptions is used. The relationships between the strains and the forces in the orthotropic materials have been defined; these relationships are not yet defined for the reinforced concrete. Such definitions must precede any investigations into the inferences as possible explanations.

CHAPTER V

A NEW CRITERION

In addition to the modification of Kirchoff's assumptions proposed in the previous chapter, an adequate model for the yield criterion of a simply reinforced concrete slab requires constitutive equations for the concrete and the steel, assumptions about the behavior of reinforcing bars in a crack-line (the kinking question) and a means of identifying the onset of yielding.

Identifying the Onset of Yielding

A yield criterion has been described as a mathematical function that defines the onset of the plastic region (yielding). A perfectly plastic region is one in which the generalized stress is independent of the generalized strain. When actually measured in tests on reinforced concrete slabs, the relationship between the generalized stress (moment) and the generalized strain (curvature) does not display a perfectly plastic region. It does, for certain small percentages of reinforcement, display a region in which the increase in moment is small for large increases in curvature, hereinafter referred to as an imperfectly plastic region. The moment reaches a maximum within this region and sometimes starts to decrease prior to a crushing of the concrete. The maximum moment attained is referred to as the ultimate moment. A plot of a typical moment-curvature relationship for a reinforced concrete slab when the steel is normal to the crack-line is shown by the solid line in Figure 10.

If the imperfectly plastic region is to be approximated by a perfectly plastic region, then a yield criterion may predict any moment that lies within the imperfectly plastic range (the upper and lower moment limits are indicated by horizontal dotted lines in the typical plot of Figure 10); this *moment, by definition, identifies* the onset of the perfectly plastic region. In this model the basis for predicting a moment in the imperfectly plastic range is borrowed from ultimate strength theory for beams, namely that the ultimate moment is determined by a principal concrete strain reaching a limiting value. The use of a limiting concrete strain enables one to determine from the model whether or not the reinforcement has yielded prior to reaching the limiting strain. Such knowledge is important since the yielding of the reinforcement is the physical explanation for the existence of an imperfectly plastic region in the moment-curvature relationship. Furthermore, ultimate strength theory for beams predicts the ultimate moment in the imperfectly plastic region; consequently, it is the ultimate moment that the new criterion claims to predict.

Heretofore it has been assumed that the ultimate moment is determined by a limiting compressive principal strain. The use of the modified Kirchoff's assumptions raises the possibility of a limiting tensile principal strain at the top of the crack. This possibility must be taken into consideration when determining the ultimate moment. In order to do this it is necessary to distinguish between a micro-crack, which is an initial crack at a point, and the macro-crack of the yield-line, which is part of the collapse pattern of the slab.

It is assumed that the micro-crack can occur at any orientation to the macro-crack, after which it reorients to coincide with the macro-crack as the collapse pattern forms.

Constitutive Equations for the Concrete

The stress-strain relationship for concrete obtained from a uniaxial compression test is non-linear, and of even greater importance is the fact that the stress goes through a maximum prior to reaching the rupture strain of the concrete. Well-defined and accepted transformations to any other arbitrary plane for such a stress-strain relationship could not be found in the available literature. Fortunately, an accurate description of these transformations is not critical in achieving the objective of this thesis, since the desired quantities result from integrals involving the stress-strain relationship. As long as the values predicted by the integration process agree with experimental results, the particular form of stress-strain relationship used is unimportant. A good example of such an idealization is the rigid-plastic stress-strain relationship for concrete which is used in ultimate strength theory for beams, i.e., the rectangular stress block method.

A linear elastic, perfectly plastic (hereinafter referred to as elastic-plastic) stress-strain relationship is used herein as a model for the concrete. The transformations for the linear elastic range to any other arbitrary plane are well defined and accepted, and several approaches are available for the plastic range. As previously stated, the rupture of the material is assumed to occur at some particular value of principal strain, one value for compression and another value for tension. A typical elastic-plastic stress-strain relationship is shown graphically by the solid lines in Figure 11; a typical relationship obtained from a uniaxial compression test is shown by the dashed line.

When the crack-line is perpendicular to the reinforcement and the moment is a principal one, the elastic-plastic model should predict the same ultimate moment in a slab as the current rigid-plastic model for beams. This requirement is used to determine the properties of an equivalent elastic-plastic model for the concrete. The predicted ultimate moments for each model will be equal if, for the same distance to the neutral axis and the same value of limiting strain, the resultant force in the concrete and the moment of that force about the neutral axis are the same in both models. In this way a determination can be made of the following properties of the equivalent elastic-plastic model:

E_c = elastic modulus

σ_{pc} = plastic stress

ε_{pc} = strain at which plastic region begins.

A more detailed explanation of their determination is presented in Appendix A.

In applying this idealized description of the behavior of concrete to a particular problem the generalized Hooke's law in terms of strains is used, from which follows

$$
\begin{aligned}
e_{nn} &= \varepsilon_{nn} - \nu\,\varepsilon_{tt} \\
e_{tt} &= \varepsilon_{tt} - \nu\,\varepsilon_{nn} \\
e_{11} &= \varepsilon_{11} - \nu\,\varepsilon_{22} \\
e_{22} &= \varepsilon_{22} - \nu\,\varepsilon_{11}
\end{aligned}
\tag{48}
$$

and from the strain invariants follows

$$
\begin{aligned}
e_{11} + e_{22} &= e_{nn} + e_{tt} \\
e_{11}e_{22} &= e_{nn}e_{tt} - (\gamma_{nt}/2)^2
\end{aligned}
\tag{49}
$$

where

ν = Poisson's ratio for concrete

e_{nn} = total strain in the n-direction

e_{tt} = total strain in the t-direction

e_{11} = total strain in the major principal strain direction

e_{22} = total strain in the minor principal strain direction

γ_{nt} = total shearing strain on n-t planes

ε_{11} = uniaxial strain in the i-direction; i = 1,2,n,t.

When the strain is in the linear portion of the idealized stress-strain relationship, the shearing stress is given by

$$
\tau_{nt} = \frac{E_c \gamma_{nt}}{2(1 + \nu)} \tag{50}
$$

When the strain is in the plastic portion, the shearing stress remains constant; furthermore, if equations (48) are to be used in this region, Poisson's ratio should be permitted to vary (21). For reinforced concrete slabs, however, the variation in Poisson's ratio has a very small effect on the predicted ultimate moment. Using the new yield criterion and Lenkei's data, when Poisson's ratio was varied from zero to its maximum of 0.5, the largest variation in the predicted ultimate moment was less than one percent; consequently, Poisson's ratio is assumed to be constant for both the linear and the plastic regions.

Behavior of Reinforcing Bars in a Crack-Line

Figure 12a shows a reinforcing bar crossing a crack-line in a typical segment of slab as one would see it if the concrete were transparent; the shaded portion indicates the plane that is shown in plan-view in Figures 12b and 13. Figure 12b shows the crack-line before any movement occurs and indicates the location of a hypothetical cut through the reinforcing bar. Figure 13 shows the crack-line after the concrete portions on each side of the crack have moved apart through a distance dN and shifted along the crack-line through a distance dT. The two faces of the hypothetical cut must be rejoined to restore continuity to the bar. They must move through a distance dY parallel to the axis of the bar and through a distance dX normal to the axis of the bar; they may also have to rotate.

Several idealizations regarding the behavior of a reinforcing bar crossing a crack-line were mentioned previously when discussing current yield criteria. The most commonly used idealization assumes that the force in the bar remains uniaxial, i.e., the effect of kinking is negligible. As seen by referring to Figure 13, this idealization has the following effect:

(1) the faces of the hypothetical cut are assumed not to rotate

(2) the faces of the hypothetical cut move through the distance dY

(3) the movement of the faces of the hypothetical cut through the distance dX is neglected.

Using this idealization it is possible to express the axial strain in the bar in terms of the strains in the n and t-directions and the angle α. Letting

Y = initial length of bar

dY = change in length of bar

ε_y = axial strain in bar

and referring to Figure 13, the following relationships can be observed

$$Y = N/\sin\alpha \tag{51}$$

$$dY = dN \sin\alpha + dT \cos\alpha \ .$$

Then, by definition

$$\varepsilon_y = dY/Y \tag{52}$$

and substituting equations (51) into this gives

$$\varepsilon_y = (dN/N)\ \sin^2\alpha + (dT/N)\ \sin\alpha\cos\alpha \ . \tag{53}$$

Realizing that the terms in parentheses in equations (53) are the strains in the n and t-directions respectively, the axial strain in the bar becomes

$$\varepsilon_y = \varepsilon_{nn} \sin^2\alpha + \gamma_{nt} \sin\alpha \cos\alpha \ . \tag{54}$$

Obtained in a similar manner, the axial strain in a bar parallel to the x-axis is

$$\varepsilon_x = \varepsilon_{nn} \cos^2\alpha - \gamma_{nt} \sin\alpha \cos\alpha \ . \tag{55}$$

Use of the uniaxial bar force idealization also enables one to use the stress-strain relationship obtained from a simple tension test as the constitutive equation for the steel, consequently the stress in the bar can be expressed in terms of the crack-line strains for any arbitrary orientation of the bar axis to the crack-line.

The uniaxial force idealization, as expressed by equations (54) and (55), along with the stress-strain relationship obtained from a simple tension test of a steel bar are used in the new criterion.

A New Criterion

A yield criterion for reinforced concrete slabs is currently understood to be a means of predicting, for any given orientation of a crack-line to the reinforcement, a value of bending moment per unit width of cross-section along the crack-line which corresponds to the onset of yielding. As mentioned previously, the onset of yielding can be defined in different ways; for purposes of comparison with tests, it is best defined as the ultimate moment.

An ultimate bending moment, along with any twisting moment that occurs, on a particular cross-section along a crack-line can be resolved into an equivalent system of forces with components normal to and parallel to the crack-line; furthermore, the equivalent system of forces can be resolved into an equivalent stress distribution on the cross-section. Any scheme, then, that provides a stress distribution on the cross-section of interest which results in a computed moment that gives acceptable agreement* with ultimate bending moments measured in tests, meets the currently understood definition of a yield criterion for reinforced concrete slabs.

In the new criterion, the scheme for providing a stress distribution which results in an ultimate bending moment utilizes the maximum principal strain theory of failure and applies it to the concrete. Any strain distribution on a cross-section along a crack-line which is in accordance with the modified Kirchoff's assumptions and results in either the principal compressive strain on the surface of the slab or the principal tensile strain at the neutral axis reaching a limiting value is one whose corresponding stress distribution results in an ultimate bending moment.

In order to predict the collapse load of a slab, a yield criterion is generally used in conjunction with a flow rule and the upper and lower-bound theorems of limit analysis. No flow rule is associated with the new criterion; instead, the constitutive equations are assumed to be valid up to the attainment of an ultimate bending moment. In this way it is possible to retain the ingredients of a classical plate theory:

(1) constitutive relations for the materials

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*<u>Acceptable agreement</u> from an engineering point of view generally implies that the predicted ultimate bending moment be equal to or less than the measured - equal being the ideal. When a statistical approach is used, a certain number of measured moments less than predicted moments are acceptable.

(2) an assumption about the geometry of deformation

(3) satisfaction of the equilibrium conditions

(4) satisfaction of the boundary conditions.

Summary

The new criterion predicts the moment at the upper limit of the imperfectly plastic region, i.e., the ultimate moment. Since it is precisely defined and easily measured, it is a better basis for comparing predicted and reported experimental results than the moment at the lower limit. The moment at the lower limit may be defined differently by each investigator, and the value obtained is dependent upon the technique employed to extract it from the measured data.

The new criterion employs elastic-plastic idealizations for the uniaxial stress-strain behavior of both the concrete and the steel. The concrete has upper and lower limits on its strain values; it is considered to have ruptured whenever these values are exceeded. No strain limits are set on the steel, since a steel rupture prior to a concrete rupture is highly unlikely. Consequently, the ultimate moment is reached when the concrete can accept no more load without rupturing.

The effect of "kinking" is neglected in the new criterion.

In applying the new criterion to a particular slab, the modification of Kirchoff's assumptions is taken as valid.

CHAPTER VI

ATTAINMENT OF OBJECTIVE

A difference in behavior can be observed by comparing the results of uniaxial bending tests on reinforced concrete slabs reported by Lenschow and Sozen (14) with those reported by Lenkei (15). The difference appears in tests on orthotropically reinforced slabs in which the yield-line is not perpendicular to the reinforcement; it is seen when the reported test values are compared with values predicted by using Johansen's criterion. Lenschow and Sozen's results reveal no significant difference between the measured values and the predicted values; many of Lenkei's measured values are significantly lower than the predicted values. A rational explanation for this difference in behavior is the objective of this dissertation.

The only essential difference between the two test set-ups occurs in the boundary conditions. Lenschow and Sozen's set-up permits twisting; Lenkei's does not. Consequently, any rational explanation must satisfy these two conditions:

(1) it must provide a physical explanation for the reduction in ultimate moment capacity of certain cross-sections when twisting is prevented

(2) it must demonstrate that the addition of twisting to these same cross-sections leads to higher values of ultimate moment, in fact to values very close to those predicted

by using Johansen's criterion.

The criterion proposed in Chapter V provides the key to such an explanation.

The first column of Table 1 lists Lenkei's tests, using his identifying numbers, on orthotropically reinforced concrete slabs in which the yield-line did not occur perpendicular to the reinforcement. A comparison between his measured values and the values predicted by using Johansen's criterion is shown in column three. The measured values are less than the predicted values with the exception of tests 33 and 34. Since these two tests are the only deviations from the general pattern, their validity should be confirmed by further testing before trying to offer any explanation for their occurrence; consequently they are ignored in what follows and only an explanation for the general pattern is offered.

The idealized version of Johansen's criterion, which is derived in Chapter II and expressed in equations (20), is not used in determining the ratios shown in columns three and four of Table 1. Instead of computing the ultimate moment per unit width for the cross-sections normal to the reinforcement and expressing the ultimate moment per unit width on any other cross-section as a function of these two moments, the ultimate moment per unit width of cross-section along the yield-line is computed directly. The same assumption for determining the ultimate moments of the cross-sections normal to the reinforcement is used in determining the ultimate moment of the cross-section along the yield-line, namely, that the actual stress distribution in the concrete can be replaced by the statically equivalent rectangular stress distribution used in ultimate strength theory for beams. This version of Johansen's criterion leads to the following equations from which the values used in preparing Table 1 were obtained

$$F_{xn} = (A_x/s_x)\ f_{px} \cos^2 \alpha \tag{56}$$

$$F_{yn} = (A_y/s_y)\ f_{py} \sin^2 \alpha$$

$$a = (F_{xn} + F_{yn})/(0.85\ f'_c)$$

$$M_{nn}^{(J)} = F_{xn}(d_x - a/2) + F_{yn}(d_y - a/2)$$

where the only new symbols are

a = depth of equivalent rectangular stress block

f_{px} = yield stress under uniaxial load of x-direction bar

f_{py} = yield stress under uniaxial load of y-direction bar

$M_{nn}^{(J)}$ = ultimate moment per unit width of cross-section along a yield line (the superscript indicates that it is a version of Johansen's criterion).

This version is used in order that the new criterion and Johansen's criterion can be compared on the same basis, since the new criterion is applied directly to the cross-section along the yield-line.

Reduced Ultimate Moment When Twisting is Absent

Implicit in the version of Johansen's criterion used in preparing Table 1 is the assumption that the location of the resultant force in the concrete is determined only by the normal force component of the resultant force in the reinforcement. In the new criterion both the normal and the shearing components of the resultant force in the reinforcement affect the location of the resultant force in the concrete; consequently, this difference is a possible explanation for the reduced values of ultimate moment which occurred in Lenkei's tests. By applying the new criterion to each of Lenkei's tests and assuming, as in ultimate strength theory for beams, that the ultimate moment occurs when the principal compressive strain at the slab's surface reaches a limiting value, expected ultimate moment values can be computed (see application (A) in Appendix B). In column four of Table 1, the results of such computations are compared with those obtained using Johansen's criterion. A perusal of column four reveals the following:

(1) the values obtained using Johansen's criterion are equal to or greater than the values obtained with the new criterion using a limiting value of compressive strain at the slab's surface

(2) the difference between any two corresponding values is less than four percent.

Obviously, predicted reductions of less than four percent can not be offered as an explanation for the measured reductions as high as 37 percent which are indicated in column three; some other explanation is required.

Observing that a state of pure shear exists at the neutral axis when the new criterion is used and twisting is absent, another possible explanation is the occurrence of a limiting value of principal tensile strain at the neutral axis prior to the attainment of a limiting value of principal compressive strain at the slab's surface. By assuming that the ultimate moment values predicted by using the new criterion are equal to the ultimate moment values obtained in Lenkei's tests, values of shearing strain at the neutral axis - an indirect measure of principal tensile strain - can be computed for each of Lenkei's tests (see application (B) in Appendix B). In column five of Table 1, the results of such computations are compared with the values of shearing strain at the neutral axis computed when assuming the attainment of a limiting value of principal compressive strain at the slab's surface, i.e., those values computed in application (A) in Appendix B. In all cases the shearing strain at the neutral axis required to attain a limiting value of principal compressive strain at the slab's surface is greater than the shearing strain at the neutral axis required to attain ultimate moments equal to those measured in Lenkei's tests. It follows, then, that the reduction in ultimate moment observed in Lenkei's tests can be attributed to the attainment of a limiting principal tensile strain at the neutral axis prior to the attainment of a limiting principal compressive strain at the slab's surface; hence, the first condition for a rational explanation (to provide a physical explanation) is satisfied.

<u>Increasing the Ultimate Moment by the Introduction of Twisting</u>

When twisting is absent, it has been shown that the ultimate moment can be determined by a limiting principal tensile strain at the neutral axis. It remains to be shown that the introduction of twisting will permit an increase in the ultimate moment, which is apparently what occurred in Lenschow and Sozen's tests.

If a limiting principal compressive strain at the slab's surface has not been attained when a limiting principal tensile strain is reached at the neutral axis, any amount of twisting which does not result in a principal compressive strain at the slab's surface exceeding a limiting value is permitted by the new criterion. By setting the principal tensile strain at the neutral axis to a limiting value and using the amount of twisting as a search-variable, the maximum value of ultimate moment attainable on the cross-section can be computed for each of Lenkei's tests (see application (C) of Appendix B). In computing these values, the shearing strain at the neutral axis corresponding to a limiting value of principal tensile strain is taken as the value required for the new criterion to predict the ultimate moment values obtained in Lenkei's tests, i.e., the shearing strain computed in Application (B) of Appendix B. In column six of Table 1, the results of these computations are compared with the results obtained by using Johansen's criterion. A glance at column six reveals that the ultimate moments obtained by introducing twisting are within one percent of those obtained by using Johansen's criterion, with the exception of test 28; and in test 28 Johansen's value is just three percent greater. It follows, then, that the lack of any significant difference in Lenschow and Sozen's tests between the measured values and the values predicted by using Johansen's criterion can be attributed to the presence of twisting; thus the second condition for a rational explanation (to demonstrate an increase in ultimate moment as a result of twisting) is satisfied.

Comments on Anomalies in Table 1

The anomalies in Table 1 deserve some comment.

The fact that the measured ultimate moments in tests 33 and 34 are considerably greater than the ultimate moments predicted by using Johansen's criterion, while all the others are considerably less, has already been mentioned. As presently structured, the new criterion is unable to account for measured ultimate moments which are greater than those predicted by using Johansen's criterion; however, some assumptions with regard to the reinforcing steel are utilized that can be changed to account for greater ultimate moments. Consideration of either strain-hardening in the steel or the kinking effect or both can lead to higher values of ultimate moment.

The other obvious anomaly occurs in test 19 and appears in column five; the ratio between the shearing strains in thise case is noticeably larger than the ratios in the other cases. Actually, the values of these ratios have very little physical significance. The physically significant values are the shearing strains computed in application (B) of Appendix B, for, at the neutral axis, they can be easily transformed into the principal tensile strains at which the concrete ruptured. Using the idealized elastic-plastic stress-strain relationship for concrete, a tensile rupture stress for the concrete can be computed. The results of such computations are compared, in column seven of Table 1, with the cylinder strengths of the concrete. Referring again to test 19, this time in column seven, the computed tensile strength of the concrete is only six-tenths of one percent of the measured compressive strength. Although this is a very low value, between 10 and 12 percent being a widely accepted norm, the occurrence of such a large deviation in the testing of concrete specimens is not unheard of. Furthermore, it must be remembered that these values of tensile rupture strength are obtained indirectly by computation from a model which utilizes assumptions that make dubious the correspondence between the computed value and the real value, the most obvious of these assumptions being the idealized elastic-plastic stress-strain relationship and the two assumptions with respect to the reinforcement mentioned in the previous paragraph. Nevertheless, if the new criterion, along with the accepted norm for concrete tensile rupture strength of 10 percent of the compressive strength, had been used to predict the ultimate moment values from Lenkei's tests, the predicted values would have been unconservative in only four cases.

Summary

The results of uniaxial bending tests on reinforced concrete slabs reported by Lenschow and Sozen reveal no significant difference between the measured values of ultimate moment and the values predicted by using Johansen's criterion. The reported results of similar tests by Lenkei reveal measured values of ultimate moment significantly less than the values predicted by Johansen's criterion for most of those slabs which are orthotropically reinforced and in which the yield-line is not perpendicular to the reinforcement. The only essential difference between Lenschow and Sozen's test set-up and Lenkei's test set-up is in the boundary conditions. Lenschow and Sozen's set-up permits twisting; Lenkei's does not.

By using the new criterion proposed in Chapter V, one is able to predict the difference in behavior which occurred in these tests.

The lower ultimate moment capacity when twisting is prevented can be attributed to the occurrence of a concrete tensile failure at the neutral axis prior to the occurrence of a concrete compressive failure at the slab's surface. The higher ultimate moment capacity when twisting is permitted can be attributed to the redistribution of stress allowed by the twisting action until simultaneous failures in the concrete occur, in tension at the neutral axis and in compression at the slab's surface.

CHAPTER VII

COMMENTS, CONCLUSIONS AND RECOMMENDATIONS

Considering the relatively small amount of experimental information utilized, only three conclusions, stated towards the end of the chapter, appear to be justifiable at the present time. Of considerable interest, however, are the many ramifications suggested by the proposed new criterion.

On the Effect of Support Conditions

In Lenkei's test set-up, the supports offer no resistance to translation in the n-t plane (the plane of the slab). If translation in the n-t plane is prevented, a beneficial effect (having nothing to do with membrane action) on the ultimate moment capacity results. When translation is allowed, the reactive force systems at the supports are twisting moments - depicted as solid arrows in Figure 15a. When translation is prevented, the reactive systems at the supports are shear forces parallel to the supports plus moments about axes perpendicular to the n-t plane - depicted as solid arrows in Figure 15b. It is the change in the nature of the reactive systems that accounts for the beneficial effect.

When translation is allowed, the resultant force system on a cross-section of the slab at some distance from and parallel to the support consists of two force couples - one normal to the cross-section in a plane perpendicular to the plane of the slab and one in the plane of the cross-section (see Figure 15c). Under these circumstances a shearing force is required in the concrete, which accounts for a lower ultimate moment capacity. When translation is prevented, the resultant force system on a cross-section of the slab at some distance from and parallel to the support consists of a force couple normal to the cross-section and a resultant force in the plane of the cross-section (see Figure 15d). The orientation of the plane of the normal force couple varies with the magnitude of moment required for equilibrium of the moments about an axis perpendicular to the plane of the slab. Under these circumstances no shearing force is required in the concrete, hence an increase in the ultimate moment capacity.

When only partial translational restraint is provided, some combination of these two resultant force systems will occur, and the magnitude of each will depend on the amount of restraint provided.

On Shear Normal to the Plane of the Slab

In the foregoing, only shearing forces parallel to the plane of the slab were considered, since they were the only type that resulted from the application of a uniaxial bending moment. Practical loadings on slabs generally include a component of shearing force normal to the plane of the slab. Since the new criterion uses a limiting principal strain approach to failure and since the maximum shearing strain normal to the plane of the slab occurs at the neutral axis, a limiting value of principal tensile strain at the neutral axis may be controlled either by the shearing strain normal to the plane of the slab or by the shearing strain parallel to the plane of the slab - whichever first causes the limiting tensile strain to occur.

On a Unified Theory of Ultimate Strength

At the present time the ultimate strength of reinforced concrete members in bending, shear, torsion, axial force and various combinations of these are, for the most part, treated as separate items - each described by its own equation composed of selected variables and empirically determined coefficients. The new criterion, perhaps with further refinements, suggests the possibility of obtaining a common basis for determining the ultimate state on a cross-section subjected to different combinations of force components. As an example, consider the case of combined bending and shear in a beam. In a manner analogous to the slabs in Lenkei's test set-up, the ultimate condition can occur either as a limiting principal compressive strain at the surface or as a limiting principal tensile strain at the neutral axis. As a further example, consider the case of pure torsion in a beam. In a manner analogous to the slabs in Lenschow and Sozen's test set-up, the cross-section on which the ultimate condition occurs can be other than the cross-section normal to the direction of span and the ultimate condition can occur as a limiting principal compressive strain, tensile strain or both. In fact, this approach to pure torsion was used,* along with Johansen's criterion, with good results.

On the Correspondence Between Surface Strains and Curvature

When using Kirchoff's assumptions, the directions of principal strain coincide with the directions of principal curvature. Under these circumstances it is possible to determine experimentally the directions

*See Lenschow and Sozen (19), p. 123.

of principal curvature by measuring the strains on the surface of the slab. When using the new criterion, which incorporates the modified Kirchoff's assumptions proposed in Chapter IV, the directions of principal strain do not necessarily coincide with the directions of principal curvature; consequently, experimental measurement of the surface strain is not sufficient to determine the directions of principal curvature.

On Practical Aspects

In its present form the new criterion requires the use of an electronic digital computer for its efficient application, even in the relatively simple case presented herein. Consequently, it is not yet a practical design tool in the sense that it is a compact expression relatively easy to apply with the aid of a calculator and design charts. It is such a tool in the sense that it makes the designer, at least one who can visualize the behavior of his structure, aware of possible problem areas when using his present design tools.

If proven by sufficient testing and further refinement to be an adequate predictor of the ultimate behavior of reinforced concrete members, the new criterion can serve as a standard against which to judge the deviations of simplified formulations devised for use in design practice. In fact, with the growing use of electronic digital computers, it could be used directly in design practice.

Conclusions

The following conclusions can be reached on the basis of the premises and experimental data utilized herein:

(1) for a rectangular slab, continuously supported along two opposite edges on simple supports offering no resistance to translation in the plane of the slab, unsupported along the other two edges and subjected to uniaxial bending, the solution from orthotropic plate theory indicates a principal curvature parallel to the supported edges; if the slab is composed of orthotropically reinforced concrete with the reinforcing steel not perpendicular to the supported edges, a condition of principal curvature parallel to the supported edges leads to a violation of the conditions of equilibrium (see Chapter IV)

(2) for the above mentioned orthotropically reinforced concrete slab, the postulation of a uniform shearing strain parallel to the supported edges will permit the conditions of equilibrium to be satisfied while maintaining a condition of principal curvature parallel to the supported edges (see Chapter IV)

(3) using the new criterion proposed in Chapter V, one is able to predict, for the above mentioned orthotropically reinforced slab in which twisting is prevented, an ultimate moment capacity lower than the predicted ultimate moment capacity of this same slab if twisting were not prevented - a fact observed when comparing the results of tests reported by Lenschow and Sozen with those reported by Lenkei (see Chapter VI).

Recommendations

Since the amount of experimental data presented in support of the new criterion is relatively small and since there are two exceptions in this data, more tests on orthotropically reinforced concrete slabs are needed to establish the viability of the new criterion. In designing the test set-up, particular attention should be given to the support conditions.

Much experimental data on beams subjected to combined bending and shear is available in the literature; consequently, the application of the new criterion to these tests and a comparison of the results is another way of measuring its viability. A similar statement, with less experimental data available, can be made about beams in pure torsion.

Other ramifications of the new criterion could be explored; until its validity is more firmly established, however, such exploration would seem to be premature.

APPENDICES

APPENDIX A

DETERMINING THE PROPERTIES OF AN ELASTIC-PLASTIC MODEL FOR CONCRETE

Figure 14 shows the strain diagram at ultimate moment, a rigid-plastic stress diagram and an elastic-plastic stress diagram on the concrete portion of a unit width cross-section of slab which is subjected to a principal moment. The symbols shown in this figure are defined as follows:

f'_c = uniaxial compressive strength of a 6" x12" cylinder

k_1 = an empirically determined constant that relates the depth of the rigid-plastic stress diagram to the distance from the neutral axis to the point of ultimate compressive strain

c = distance from the neutral axis to the point of ultimate compressive strain

ε_{uc} = ultimate compressive strain

z_p = distance from neutral axis to compressive yield strain in the elastic-plastic model

ε_{pc} = compressive yield strain

σ_{pc} = compressive plastic stress

C_{rp} = resultant compressive force of rigid-plastic model

C_{ep} = resultant compressive force of elastic-plastic model.

Since the cross-section is of unit width, the resultant forces are given by the areas of the respective stress diagrams, hence

$$C_{rp} = 0.85\, f'_c\, k_1\, c \tag{57}$$

$$C_{ep} = \sigma_{pc}\, (c - z_p/2)\ .$$

The moments of these resultant forces about the neutral axis are given by

$$M_{rp} = 0.85\ f_c'\, k_1 c^2 (1 - k_1/2) \tag{58}$$

$$M_{ep} = \sigma_{pc}(c^2/2 - z_p^2/6)\ .$$

If these two models are to give the same ultimate moment, then both their moments and their resultant forces must be equal; therefore

$$0.85\ f_c'\, k_1 c = \sigma_{pc}(c - z_p/2) \tag{59}$$

$$0.85\ f_c'\, k_1 c^2 (1 - k_1/2) = \sigma_{pc}(c^2/2 - z_p^2/6)\ .$$

Now, by similar triangles from the strain diagram

$$z_p = (\varepsilon_{pc}/\varepsilon_{uc})\ c \tag{60}$$

and substitution into equations (59) gives

$$0.85\ f_c'\, k_1 c = \sigma_{pc}(1 - \varepsilon_{pc}/2\varepsilon_{uc})\ c \tag{61}$$

$$0.85\ f_c'\, k_1 c^2 (1 - k_1/2) = \sigma_{pc}\ c^2 (\tfrac{1}{2} - \varepsilon_{pc}^2/6\varepsilon_{uc}^2)$$

which, after dividing the first equation by c and the second by c^2, reduces to

$$0.85\ f_c'\, k_1 = \sigma_{pc}\ (1 - \varepsilon_{pc}/2\varepsilon_{uc}) \tag{62}$$

$$0.85\ f_c'\, k_1 (1 - k_1/2) = \sigma_{pc}(\tfrac{1}{2} - \varepsilon_{pc}^2/6\varepsilon_{uc}^2)\ .$$

Accepting the expressions

$$k_1 = 0.85 \qquad \text{for } f_c' \leq 4 \text{ ksi} . \tag{63}$$

$$k_1 = 0.85 - 0.05\,(f_c' - 4) \qquad \text{for } f_c' \geq 4 \text{ ksi} .$$

$$\varepsilon_u = 0.003$$

from ACI Standard 318-63 of the American Concrete Institute as valid, values of σ_{pc} and ε_{pc} can be determined from equations (62) and (63) for any known value of cylinder strength; then the modulus of elasticity can be determined from

$$E_c = \sigma_{pc}/\varepsilon_{pc} . \tag{64}$$

It should be noted that this is only one of several rational ways to arrive at values of σ_{pc}, ε_{pc} and E_c for the elastic-plastic model. For the purposes of this thesis, the elastic-plastic form of the stress-strain curve and correspondence with the rigid-plastic model at ultimate moment was of more importance than the particular values of σ_{pc}, ε_{pc} and E_c. For other purposes, the way in which these values are determined could be significant, e.g. if one wished to establish values of concrete tensile strength by applying the new criterion to Lenkei's tests or other similar tests.

APPENDIX B

APPLICATIONS OF THE NEW YIELD CRITERION TO LENKEI'S TEST SET-UP

A summary of the basic equations involved is presented to facilitate following their conversion into the set of equivalent equations used in each particular application. The basic equations are arranged in groups so that the source of each group can be identified. As a further aid, each equation is identified by a special code rather than simply by a number. This code has the form

$$(X.Y.Z)$$

where X is either a Roman numeral or a letter of the alphabet and Y and Z are numbers. The Roman numeral indicates that the equation is one of the basic set and identifies the group to which it belongs; the letter of the alphabet indicates that the equation arises from a particular application and identifies the application. The number Y indicates that the equation is the Y^{th} one in the set, and the number Z indicates that this is the Z^{th} modified form of this equation in the presentation.

For the meaning of a particular symbol in any of the basic equations, refer to the Nomenclature; and for a plan of the slab showing the coordinate directions, see Figure 6.

The basic equations are

Group I

$$\sigma_c = 0 \qquad \text{for } \varepsilon_c > \varepsilon_{tc} \text{ or } \varepsilon_c < -\varepsilon_{uc} \tag{I.1.0}$$

$$\sigma_c = E_c\varepsilon_c \qquad \text{for } -\varepsilon_{pc} \leq \varepsilon_c \leq \varepsilon_{tc} \tag{I.2.0}$$

$$\sigma_c = -\sigma_{pc} \qquad \text{for } -\varepsilon_{uc} \leq \varepsilon_c \leq -\varepsilon_{pc} \tag{I.3.0}$$

Group II

$$\sigma_s = \sigma_{ps} \qquad \text{for } \varepsilon_s \geq \varepsilon_{ps} \tag{II.1.0}$$

$$\sigma_s = E_s\varepsilon_s \qquad \text{for } -\varepsilon_{ps} \leq \varepsilon_s \leq \varepsilon_{ps} \tag{II.2.0}$$

$$\sigma_s = -\sigma_{ps} \qquad \text{for } \varepsilon_s \leq -\varepsilon_{ps} \tag{II.3.0}$$

Group III

$$e_{nn} = (\partial^2 w/\partial n^2)z \qquad = \beta z \tag{III.1.0}$$

$$e_{tt} = (\partial^2 w/\partial t^2)z \qquad = \psi z \tag{III.2.0}$$

$$\gamma_{nt} = 2(\partial^2 w/\partial n\partial t)z + k_{nt} = \rho z + \gamma_o \tag{III.3.0}$$

Group IV

$$e_{nn} = \varepsilon_{nn} - \nu\varepsilon_{tt} \tag{IV.1.0}$$

$$e_{tt} = \varepsilon_{tt} - \nu\varepsilon_{nn} \tag{IV.2.0}$$

$$e_{11} = \varepsilon_{11} - \nu\varepsilon_{22} \tag{IV.3.0}$$

$$e_{22} = \varepsilon_{22} - \nu\varepsilon_{11} \tag{IV.4.0}$$

Group V

$$e_{11} + e_{22} = e_{nn} + e_{tt} \qquad \text{(V.1.0)}$$

$$e_{11}\, e_{22} = e_{nn}\, e_{tt} - (\gamma_{nt}/2)^2 \qquad \text{(V.2.0)}$$

Group VI

$$\tau_{nt} = 0 \qquad \text{for } \varepsilon_{11} > \varepsilon_{tc} \text{ or } \varepsilon_{22} < -\varepsilon_{uc} \qquad \text{(VI.1.0)}$$

$$\tau_{nt} = (E_c\, \gamma_{nt})/2(1 + \nu) \qquad \text{for } -\varepsilon_{pc} \leq \varepsilon_{22} \text{ and } \varepsilon_{11} \leq \varepsilon_{tc} \qquad \text{(VI.2.0)}$$

$$\tau_{nt} = \tau_p \qquad \text{for } -\varepsilon_{uc} \leq \varepsilon_{22} \leq -\varepsilon_{pc} \text{ and } \varepsilon_{11} \leq \varepsilon_{tc} \qquad \text{(VI.3.0)}$$

Group VII

$$\sigma_{nn} = 0 \qquad \text{for } \varepsilon_{11} > \varepsilon_{tc} \text{ or } \varepsilon_{22} < -\varepsilon_{uc} \qquad \text{(VII.1.0)}$$

$$\sigma_{nn} = E_c \varepsilon_{nn} \qquad \text{for } -\varepsilon_{pc} \leq \varepsilon_{22} \text{ and } \varepsilon_{11} \leq \varepsilon_{tc} \qquad \text{(VII.2.0)}$$

$$\sigma_{nn} = \sigma_p \qquad \text{for } -\varepsilon_{uc} \leq \varepsilon_{22} \leq -\varepsilon_{pc} \text{ and } \varepsilon_{11} \leq \varepsilon_{tc} \qquad \text{(VII.3.0)}$$

Group VIII

$$\varepsilon_x = \varepsilon_{nn} \cos^2\alpha - \gamma_{nt} \sin\alpha \cos\alpha \qquad \text{(VIII.1.0)}$$

$$\varepsilon_y = \varepsilon_{nn} \sin^2\alpha + \gamma_{nt} \sin\alpha \cos\alpha \qquad \text{(VIII.2.0)}$$

Group IX

$$f_x = \sigma_{px} \qquad \text{for } \varepsilon_x \geq \varepsilon_{px} \qquad \text{(IX.1.0)}$$

$$f_x = E_s \varepsilon_y \qquad \text{for } -\varepsilon_{px} \leq \varepsilon_x \leq \varepsilon_{px} \qquad \text{(IX.2.0)}$$

$$f_x = -\sigma_{px} \qquad \text{for } \varepsilon_x \leq -\varepsilon_{px} \qquad \text{(IX.3.0)}$$

Group X

$$f_y = \sigma_{py} \quad \text{for } \varepsilon_y \geq \varepsilon_{py} \tag{X.1.0}$$

$$f_y = E_s \varepsilon_y \quad \text{for } -\varepsilon_{py} \leq \varepsilon_y \leq \varepsilon_{py} \tag{X.2.0}$$

$$f_y = -\sigma_{py} \quad \text{for } \varepsilon_y \leq -\varepsilon_{py} \tag{X.3.0}$$

Group XI

$$F_n = (A_y/S_y)\, f_y \,\sin^2\alpha + (A_x/S_x)\, f_x \cos^2\alpha \tag{XI.1.0}$$

$$F_t = ((A_y/S_y)\, f_y - (A_x/S_x)\, f_x) \sin\alpha \cos\alpha \tag{XI.2.0}$$

Group XII

$$F_n = -\int_{-c}^{0} \sigma_{nn}\, dz \tag{XII.1.0}$$

$$F_t = -\int_{-c}^{0} \tau_{nt}\, dz \tag{XII.2.0}$$

$$M_{nn} = \int_{-c}^{0} \sigma_{nn}\, z\, dz + F_n(d - c) \tag{XII.3.0}$$

$$M_{nt} = \int_{-c}^{0} \tau_{nt}\, z\, dz + F_t(d - c) \tag{XII.4.0}$$

Group XIII

$$e_{tt} \equiv 0 \tag{XIII.1.0}$$

Groups I and II are the idealized stress-strain relationships for concrete and steel respectively. They are approximations to data obtained in uniaxial load tests.

Group III is the modified form of Kirchoff's assumptions along with a change in symbols to facilitate writing them. The rationale behind the modification is presented in Chapter IV.

Group IV comes from the generalized Hooke's law expressed in an n-t coordinate system and in a principal strain coordinate system. Group V is the strain invariants when transforming from the n-t coordinate system to the principal strain coordinate system. These equations are found in Chapter V in the section Constitutive Equations for the Concrete.

Groups VI and VII are the constitutive equations for the concrete applied to a cross-section with an exterior normal in the n-direction. Note that the particular equation to be used is determined by the strain conditions on the principal axes.

Group VIII is the relationship between the strains in an n-t coordinate system and the uniaxial strain in a reinforcing bar expressed for bars parallel to an x-y coordinate system at some angle α to the n-t coordinate system. These equations are found in Chapter V in the section Behavior of Reinforcing Bars in a Crack-Line.

Groups IX and X are the constitutive equations for the steel applied to cross-sections with exterior normals in the x and y-directions respectively.

Group XI gives expressions for the n-direction and t-direction bar-force components per unit width of cross-section having an exterior normal in the n-direction. They are obtained from the resultant forces in the reinforcing bars as presented in Chapter II when deriving Johansen's yield criterion.

Group XII is the equilibrium equations expressed at any point (n,t) for a unit width of cross-section having an exterior normal in the n-direction.

Group XIII is an equivalent expression of the condition $\partial^2 w/\partial t^2 \equiv 0$ which is a condition present in Lenkei's test set-up.

Conversion of Basic Equations into Equivalent Forms

The original form of a set of equations is not always the best form for application in a particular case, hence the basic equations are reduced, through elimination of non-pertinent variables by substitution, to a smaller set deemed suitable for the particular applications being considered. The sequence of substitutions required to get each equation in the reduced set follows.

Replace equation (IV.1.0) with the equivalent equation (IV.1.1) obtained by substituting equation (XIII.1.0) into equation (IV.2.0) to get

$$\varepsilon_{tt} = \nu\varepsilon_{nn} \tag{IV.2.1}$$

and then substituting equation (IV.2.1) into equation (IV.1.0) to get

$$e_n = (1 - \nu^2)\varepsilon_{nn} \,. \tag{IV.1.1}$$

Replace equation (V.2.0) with equivalent equation (V.2.4) or (V.2.5). To obtain the equivalent equations, substitute equations (IV.3.0), (IV.4.0) and (XIII.1.0) into equations (V.1.0) and (V.2.0) to get

$$e_n = (1 - \nu)(\varepsilon_{11} + \varepsilon_{22}) \tag{V.1.1}$$

$$-\gamma_{nt}^2 = 4(\varepsilon_{11} - \nu\varepsilon_{22})(\varepsilon_{22} - \nu\varepsilon_{11}) \ . \tag{V.2.1}$$

Then substitute equation (IV.1.1) into equation (V.1.1) which gives

$$(1 + \nu)\varepsilon_{nn} = \varepsilon_{11} + \varepsilon_{22} \tag{V.1.2}$$

and equation (V.1.2) into equation (V.2.1) to get either

$$\gamma_{nt}^2 = 4(1 + \nu)^2(\varepsilon_{11} - \nu\varepsilon_{nn})(\varepsilon_{11} - \varepsilon_{nn}) \tag{V.2.2}$$

or

$$\gamma_{nt}^2 = 4(1 + \nu)^2(\varepsilon_{22} - \varepsilon_{nn})(\varepsilon_{22} - \nu\varepsilon_{nn}) \tag{V.2.3}$$

Now substitute equation (IV.1.1) into equation (III.1.0) and rearrange, along with a symbol change, to give

$$\varepsilon_{nn} = (\beta/(1 - \nu^2))z = \varphi z \tag{III.1.1}$$

then substitute equations (III.1.1) and (III.3.0) into equations (V.2.2) and (V.2.3) which gives the desired equations

$$\rho^2 z^2 + 2\rho\gamma_o z + \gamma_o^2 = 4(1 + \nu)^2(\varepsilon_{11}^2 - (1 + \nu)\varepsilon_{11}\varphi z + \nu(\varphi z)^2) \tag{V.2.4}$$

$$\rho^2 z^2 + 2\rho\gamma_o z + \gamma_o^2 = 4(1 + \nu)^2(\varepsilon_{22}^2 - (1 + \nu)\varepsilon_{22}\varphi z + \nu(\varphi z)^2) \ . \tag{V.2.5}$$

Now replace the pertinent equilibrium equations with equivalent ones. Substituting equations (III.1.1) and (III.3.0) into equations (VI.2.0) and (VII.2.0) gives

$$\tau_{nt} = E_c(\rho z + \gamma_o)/(2(1 + \nu)) \qquad \text{(VI.2.1)}$$

$$\text{for } -\varepsilon_{pc} \leq \varepsilon_{22} \text{ and } \varepsilon_{11} \leq \varepsilon_{tc}$$

$$\sigma_{nn} = E_c \varphi z$$

$$\text{for } -\varepsilon_{pc} \leq \varepsilon_{22} \text{ and } \varepsilon_{11} \leq \varepsilon_{tc} \qquad \text{(VII.2.1)}$$

Letting z_{pc} be the magnitude of z at which the minor principal strain has a magnitude of ε_{pc}, equations (VI.3.0) and (VII.3.0) can be rewritten as

$$\tau_{nt} = E_c(-\rho z_p + \gamma_o)/(2(1 + \nu)) \qquad \text{(VI.3.1)}$$

$$\text{for } -\varepsilon_{uc} \leq \varepsilon_{22} \leq -\varepsilon_{pc} \text{ and } \varepsilon_{11} \leq \varepsilon_{tc}$$

$$\sigma_{nn} = -E_c \varphi z_p \qquad \text{(VII.3.1)}$$

$$\text{for } -\varepsilon_{uc} \leq \varepsilon_{22} \leq -\varepsilon_{pc} \text{ and } \varepsilon_{11} \leq \varepsilon_{tc}$$

Substituting equations (VI.1.0), (VI.2.1), (VI.3.1), (VII.1.0), (VII.2.1) and (VII.3.1) into equations (XII.1.0), (XII.2.0) and (XII.3.0) along with the proper limits gives

$$F_n = \int_{-c}^{-z_p} E_c \varphi z_p \, dz + \int_{-z_p}^{0} -E_c \varphi z \, dz \qquad \text{(XII.1.1)}$$

$$F_t = \int_{-c}^{-z_p} -E_c(-\rho\ z_p + \gamma_o)/(2(1+\nu))dz + \tag{XII.2.1}$$

$$\int_{-z_p}^{0} -E_c(\rho z + \gamma_o)/(2(1+\nu))dz$$

$$M_{nn} = \int_{-c}^{-z_p} -E_c\,\varphi\,z_p\ zdz + \int_{-z_p}^{0} E_c\,\varphi\,z^2\,dz + \tag{XII.3.1}$$

$$F_n(d - c)$$

and performing the integration operation gives

$$F_n = (-E_c\varphi/2)(z_p^2 - 2\ z_p\ c) \tag{XII.1.2}$$

$$F_t = (-E_c/(4(1+\nu))(\rho z_p^2 - 2\rho\ z_p\ c + 2\gamma_o c) \tag{XII.2.2}$$

$$M_{nn} = (E_c\varphi/6)(3\ z_p\ c^2 - z_p^3) + F_n(d - c)\ . \tag{XII.3.2}$$

Finally, replace equations (VIII.1.0) and (VIII.2.0) with equivalent ones. Substituting equations (III.1.0) and (III.3.0) into equation (VIII.1.0) and into equation (VIII.2.0) gives

$$\varepsilon_x = \varphi z \cos^2\alpha - (\rho z + \gamma_o) \sin\alpha \cos\alpha \tag{VIII.1.1}$$

$$\varepsilon_y = \varphi z \sin^2\alpha + (\rho z + \gamma_o) \sin\alpha \cos\alpha \tag{VIII.2.1}$$

and since the steel is located at

$$z = d - c$$

these equations become

$$\varepsilon_x = \varphi(d - c)\cos^2\alpha - (\rho(d - c) + \gamma_o)\sin\alpha\cos\alpha \qquad \text{(VIII.1.2)}$$

$$\varepsilon_y = \varphi(d - c)\sin^2\alpha + (\rho(d - c) + \gamma_o)\sin\alpha\cos\alpha \qquad \text{(VIII.2.2)}$$

Application (A)

This application imposes the condition of no twisting strain and requires that the concrete minor principal strain be at its limiting value on the compressive surface of the slab (at $z = -c$), which results in the following equations:

$$\rho \equiv 0 \qquad \text{(A.1.0)}$$

$$(\varepsilon_{22})_{z=-c} = -\varepsilon_{uc} \qquad \text{(A.2.0)}$$

$$-\varepsilon_{pc} = (\varepsilon_{22})_{z=-z_p} \; . \qquad \text{(A.3.0)}$$

Continuing the reduction to a smaller set, substitution of equation (A.1.0) into equation (V.2.5) gives

$$\gamma_o^2 = 4(1 + \nu)^2\,(\varepsilon_{22}^2 - (1 + \nu)\varepsilon_{22}\varphi\, z + \nu(\varphi z)^2) \qquad \text{(V.2.6)}$$

and substitution of equations (A.2.0) and (A.3.0) into equation (V.2.6) gives

$$\gamma_o^2 = 4(1 + \nu)^2(\varepsilon_{uc}^2 - (1 + \nu)\varepsilon_{uc}\varphi\, c + \nu(\varphi c)^2) \qquad \text{(V.2.7)}$$

and

$$\gamma_o^2 = 4(1 + \nu)^2\,(\varepsilon_{pc}^2 - (1 + \nu)\varepsilon_{pc}\varphi\, z_p + \nu(\varphi\, z_p)^2) \, . \qquad \text{(V.2.8)}$$

Also, substitution of equation (A.1.0) into equations (VIII.1.2) and (VIII.2.2) gives

$$\varepsilon_x = \varphi(d - c)\cos^2\alpha - \gamma_o \sin\alpha\cos\alpha \tag{VIII.1.3}$$

$$\varepsilon_y = \varphi(d - c)\sin^2\alpha + \gamma_o \sin\alpha\cos\alpha \tag{VIII.2.3}$$

and into equation (XII.2.2) gives

$$F_t = (-E_c\gamma_o c)/(2(1 + \nu))\ . \tag{XII.2.3}$$

Application (A) is now reduced to the equations and variables listed in Table 2. This set, consisting of 11 equations and 11 unknowns, was solved by iteration techniques using an electronic digital computer to obtain the values of $M_{nn}^{(A)}$ and $\gamma_o^{(A)}$ used in Table 1.

Application (B)

This application imposes the condition of no twisting strain and uses Lenkei's test values for the yield moments, which results in the following equations

$$\rho \equiv 0 \tag{B.1.0}$$

$$M_{nn} = M_{nn}^{(Lenkei)}\ . \tag{B.2.0}$$

Then, assuming that the idealized concrete will be in the plastic region when the yield moment occurs provides the following equation

$$-\varepsilon_{pc} = (\varepsilon_{22})_{z=-z_p}\ . \tag{B.3.0}$$

Since equations (B.1.0) and (B.3.0) are the same as equations (A.1.0) and (A.3.0), all of the equations in application (A) are valid for application (B) except equation (V.2.7), and it is replaced by equation (B.2.0). Hence, application (B) is reduced to the list of equations and variables in Table 3. This set, consisting of 11 equations and 11 unknowns was solved by iteration techniques using an electronic digital computer to obtain the values of $\gamma_o^{(B)}$ used in Table 1.

Application (C)

This application permits a twisting strain. It also uses the values of γ_o obtained in application (B) as limiting values of shearing strain at the top of the crack (at z = 0), which results in the following equation:

$$\gamma_o = \gamma_o^{(B)} \qquad (C.1.0)$$

Again, assuming that the idealized concrete will be in the plastic region when the yield moment occurs provides the following equation:

$$-\varepsilon_{pc} = (\varepsilon_{22})_{z=-z_p} \; . \qquad (C.2.0)$$

Continuing the reduction to a smaller set, substitution of equation (C.2.0) into equation (V.2.5) gives

$$(\rho z_p)^2 - 2\rho z_p \gamma_o + \gamma_o^2 = \qquad (V.2.9)$$

$$4(1 + \nu)^2(\varepsilon_{pc}^2 - (1 + \nu)\varepsilon_{pc}\varphi z_p + \nu(\varphi z_p)^2) \, .$$

At this point application (C) is reduced to the equations and unknown variables listed in Table 4. This set of 11 equations and 12 unknowns was solved for a maximum value of M_{nn} by search techniques using an electronic digital computer. These are the values of $M_{nn}^{(C)}$ used in Table 1.

Comment

The tables listing the equations and unknown variables for each application list the equations by code number or by group number. For easy reference, Table 5 provides a listing of the equations themselves.

Table 1. Comparisons of Computed and Measured Results

Test	α°	$M_{nn}^{(J)}/M_{nn}^{(L)}$	$M_{nn}^{(J)}/M_{nn}^{(A)}$	$\gamma_o^{(A)}/\gamma_o^{(B)}$	$M_{nn}^{(J)}/M_{nn}^{(C)}$	f_{ct}/f_c'
18	67.5	1.103	1.003	4.227	1.001	0.050
19	67.5	1.168	1.005	40.292	1.001	0.006
20	67.5	1.100	1.006	2.799	1.002	0.091
21	67.5	1.063	1.008	1.538	1.003	0.180
22	45.0	1.148	1.004	2.328	0.999	0.112
23	45.0	1.126	1.004	2.368	1.000	0.104
24	45.0	1.142	1.007	1.946	1.001	0.137
25	45.0	1.056	1.007	1.579	1.001	0.164
26	22.5	1.168	1.000	1.816	0.999	0.069
27	22.5	1.078	1.002	1.108	1.000	0.137
28	22.5	1.110	1.033	1.193	1.031	0.135
33	67.5	0.883	---	---	---	---
34	67.5	0.920	---	---	---	---
35	45.0	1.146	1.007	2.048	1.000	0.165
36	45.0	1.167	1.011	1.717	1.001	0.202
37	22.5	1.200	1.003	1.813	1.000	0.103
38	22.5	1.111	1.002	1.760	0.999	0.105
41*	67.5	1.313	1.012	5.929	1.000	0.070
42*	45.0	1.370	1.012	2.535	1.000	0.131
43*	45.0	1.285	1.015	1.824	1.001	0.196
44*	22.5	1.194	1.003	1.378	0.999	0.139

*Lightweight aggregate concrete

Table 2. Application (A)

List of Equations* and Unknown Variables

	Equations	Variables
1.	(V.2.7)	γ_o
2.	(V.2.8)	φ
3.	(VIII.1.3)	c
4.	(VIII.2.3)	z_p
5.	Group IX**	ε_x
6.	Group X**	ε_y
7.	(XI.1.0)	f_x
8.	(XI.2.0)	f_y
9.	(XII.1.2)	F_n
10.	(XII.2.3)	F_t
11.	(XII.3.2)	M_{nn}

*The equations are listed by code number or by group number; for easy reference, Table 5 provides a listing of the equations themselves.

**A listing by group number indicates that the equation may be one of several different equations; the particular equation used is dependent upon the value of one of the unknown variables.

Table 3. Application (B)

List of Equations* and Unknown Variables

	Equations	Variables
1.	(B.2.0)	γ_o
2.	(V.2.8)	φ
3.	(VIII.1.3)	c
4.	(VIII.2.3)	z_p
5.	Group IX**	ε_x
6.	Group X**	ε_y
7.	(XI.1.0)	f_x
8.	(XI.2.0)	f_y
9.	(XII.1.2)	F_n
10.	(XII.2.3)	F_t
11.	(XII.3.2)	M_{nn}

*The equations are listed by code number or by group number; for easy reference, Table 5 provides a listing of the equations themselves.

**A listing by group number indicates that the equation may be one of several different equations; the particular equation used is dependent upon the value of one of the unknown variables.

Table 4. Application (C)

List of Equations* and Unknown Variables

	Equations	Variables
1.	(C.1.0)	γ_o
2.	(V.2.9)	φ
3.	(VIII.1.2)	c
4.	(VIII.2.2)	z_p
5.	Group IX**	ε_x
6.	Group X**	ε_y
7.	(XI.1.0)	f_x
8.	(XI.2.0)	f_y
9.	(XII.1.2)	F_n
10.	(XII.2.2)	F_t
11.	(XII.3.2)	M_{nn}
12.		ρ

*The equations are listed by code number or by group number; for easy reference, Table 5 provides a listing of the equations themselves.

**A listing by group number indicates that the equation may be one of several different equations; the particular equation used is dependent upon the value of one of the unknown variables.

Table 5. List of Equations Used in Applications (A), (B), and (C)

	$\gamma_o^2 = 4(1+\nu)^2\,(\varepsilon_u^2 - (1+\nu)\varepsilon_u\,\varphi\,c + \nu(\varphi c)^2)$	(V.2.7)
	$\gamma_o^2 = 4(1+\nu)^2\,(\varepsilon_{pc}^2 - (1+\nu)\varepsilon_{pc}\,\varphi\,z_p + \nu(\varphi z_p)^2)$	(V.2.8)
	$(\rho z_p)^2 - 2\rho z_p\gamma_o + \gamma_o^2 = 4(1+\nu)^2(\varepsilon_{pc}^2 - (1+\nu)\varepsilon_{pc}\,\varphi\,z_p + \nu(\varphi\, z_p)^2)$	(V.2.9)
	$\varepsilon_x = \varphi(d-c)\cos^2\alpha - (\rho(d-c)+\gamma_o)\sin\alpha\cos\alpha$ $\varepsilon_y = \varphi(d-c)\sin^2\alpha + (\rho(d-c)+\gamma_o)\sin\alpha\cos\alpha$	(VIII.1.2)
	$\varepsilon_x = \varphi(d-c)\cos^2\alpha - \gamma_o\sin\alpha\cos\alpha$	(VIII.1.3)
	$\varepsilon_y = \varphi(d-c)\sin^2\alpha + \gamma_o\sin\alpha\cos\alpha$	(VIII.2.3)
	$f_x = \sigma_{px}$ for $\varepsilon_x \geq \varepsilon_{px}$	(IX.1.0)
<u>Group IX</u>	$f_x = E_s\varepsilon_x$ for $-\varepsilon_{px} \leq \varepsilon_x \leq \varepsilon_{px}$	(IX.2.0)
	$f_x = -\sigma_{px}$ for $\varepsilon_x \leq -\varepsilon_{px}$	(IX.3.0)
	$f_y = \sigma_{py}$ for $\varepsilon_y \geq \varepsilon_{py}$	(X.1.0)
<u>Group X</u>	$f_y = E_s\varepsilon_y$ for $-\varepsilon_{py} \leq \varepsilon_y \leq \varepsilon_{py}$	(X.2.0)
	$f_y = -\sigma_{py}$ for $\varepsilon_y \leq -\varepsilon_{py}$	(X.3.0)
	$F_n = (A_y/S_y)f_y\sin^2\alpha + (A_x/S_x)f_x\cos^2\alpha$	(XI.1.0)
	$F_t = ((A_y/S_y)f_y - (A_x/S_x)f_x)\sin\alpha\cos\alpha$	(XI.2.0)

Table 5. (Continued)

$F_n = (-E_c\varphi/2)(z_p^2 - 2\, z_p\, c)$	(XII.1.2)
$F_t = (-E_c/(4(1 + \nu))(\rho z_p^2 - 2\rho\, z_p\, c + 2\gamma_o\, c)$	(XII.2.2)
$M_{nn} = (E_c\varphi/6)(3\, z_p\, c^2 - z_p^3) + F_n(d - c)$	(XII.3.2)
$F_t = (-E_c\gamma_o c)/(2(1 + \nu))$	(XII.2.3)
$M_{nn} = M_{nn}^{(Lenkei)}$	(B.2.0)
$\gamma_o = \gamma_o^{(B)}$	(C.1.0)

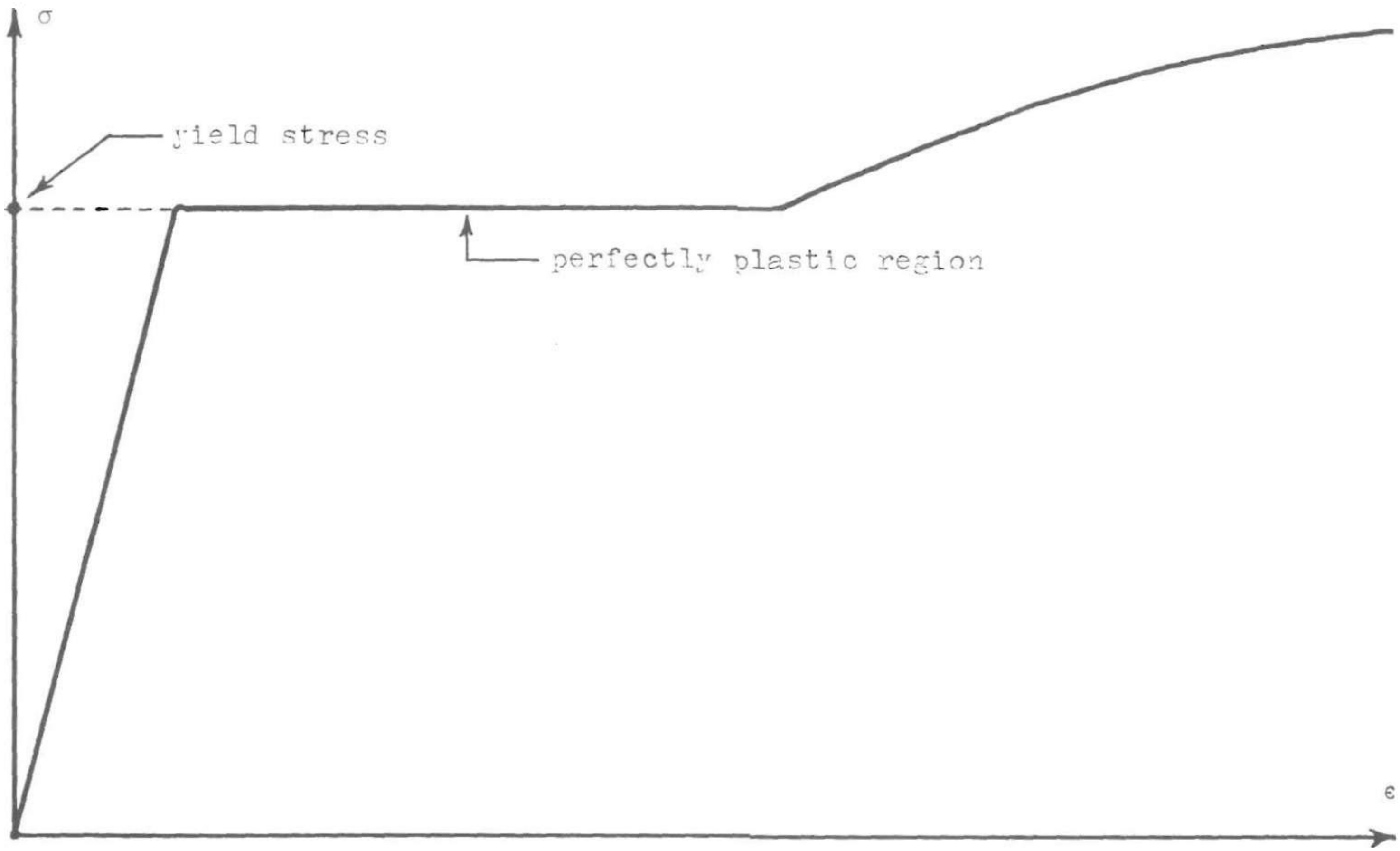


Figure 1. Portion of Typical, Tensile, Stress-Strain Curve for Mild Steel.

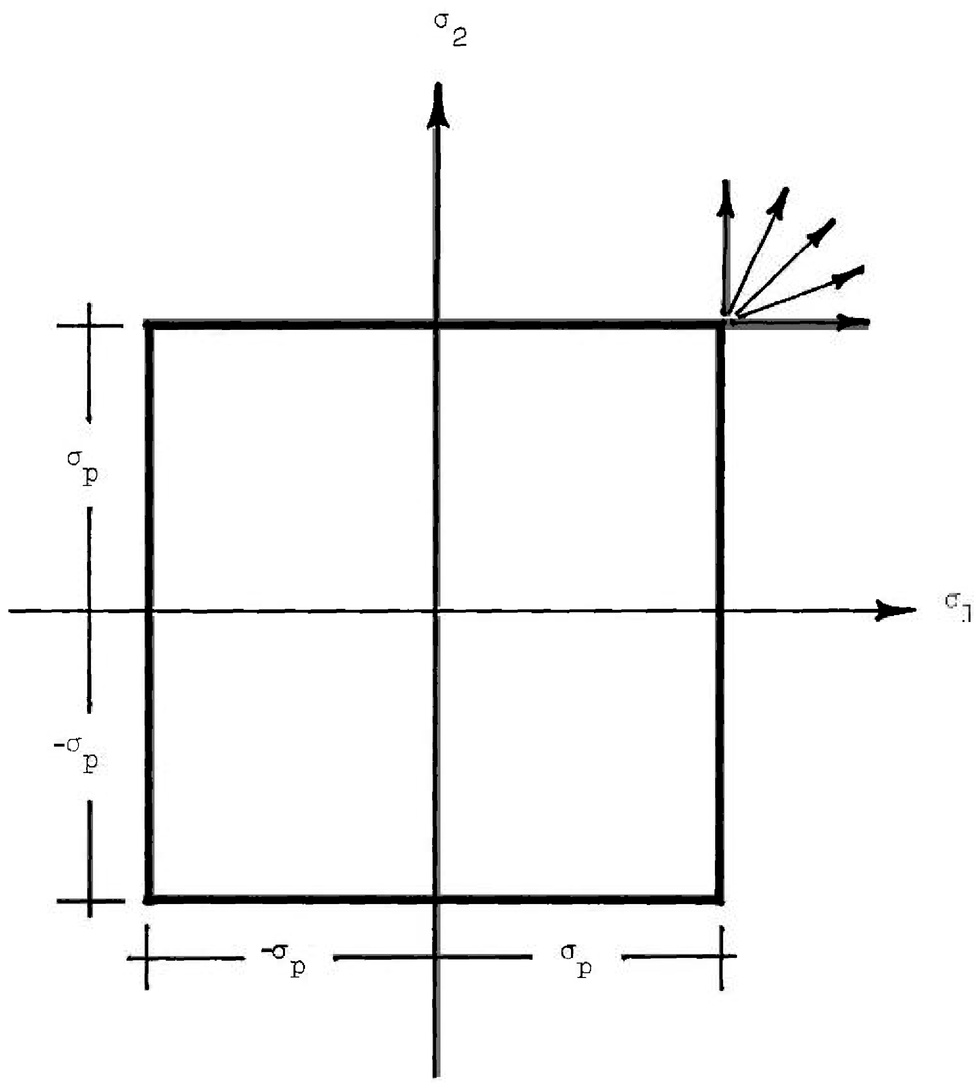


Figure 2. Example of Yield Criterion Using Maximum Principal Stress Theory.

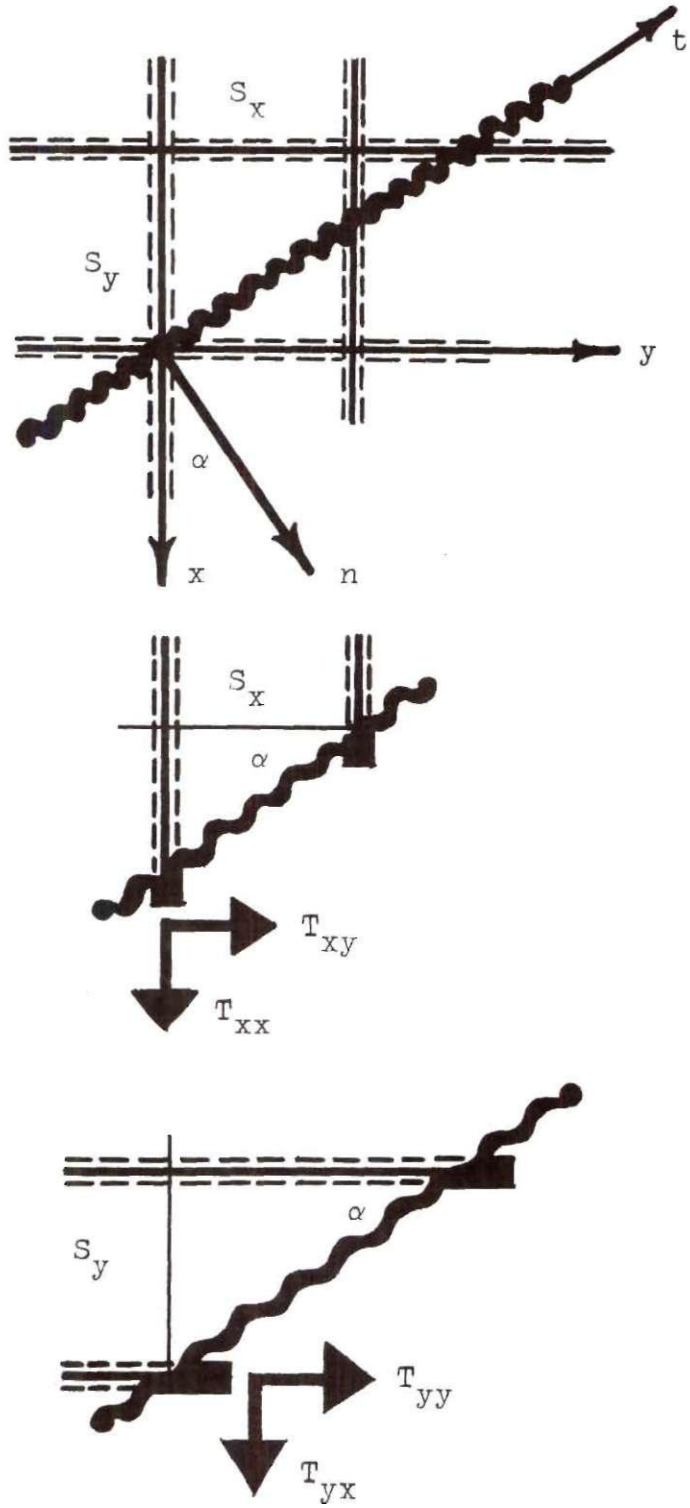


Figure 3. Relationships between Bar Forces and Length of Crack-Line for Conversion to Equivalent Line Forces.

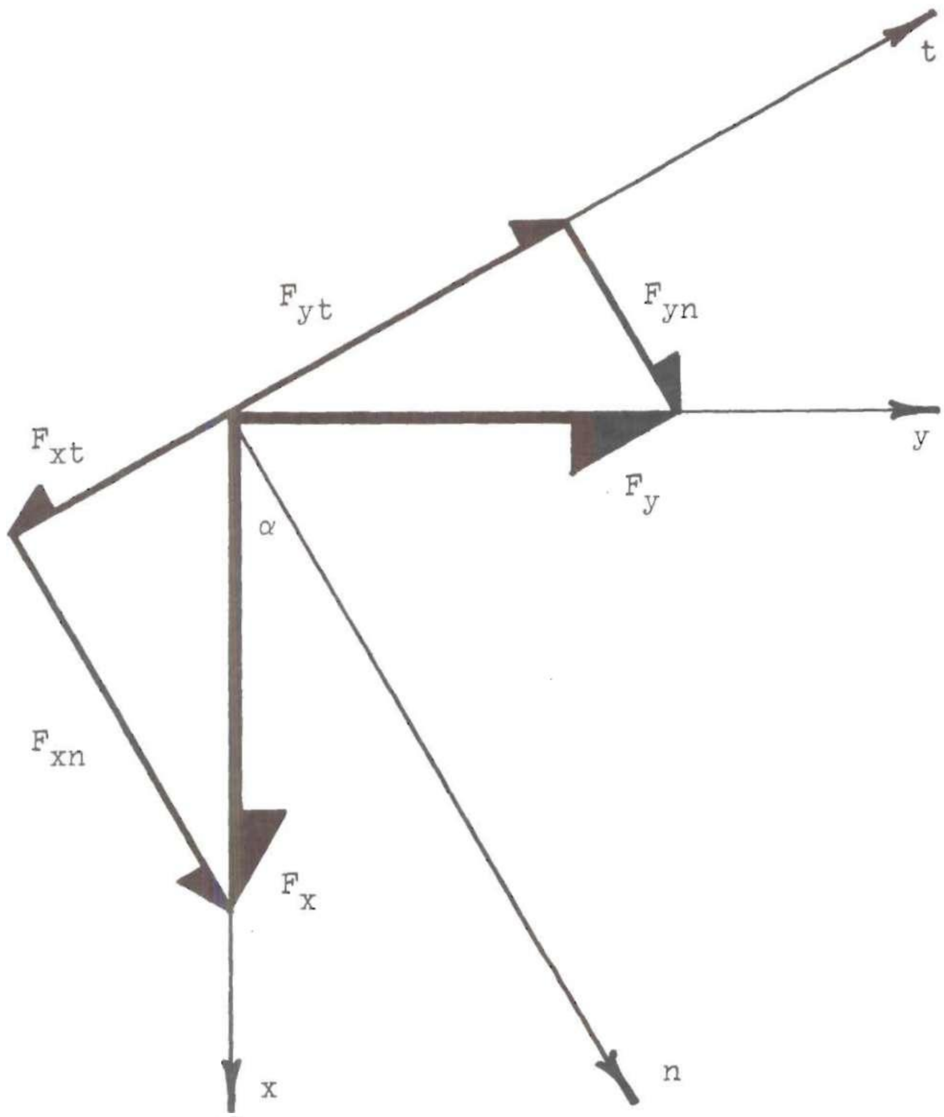


Figure 4. Components of Equivalent Line-Forces.

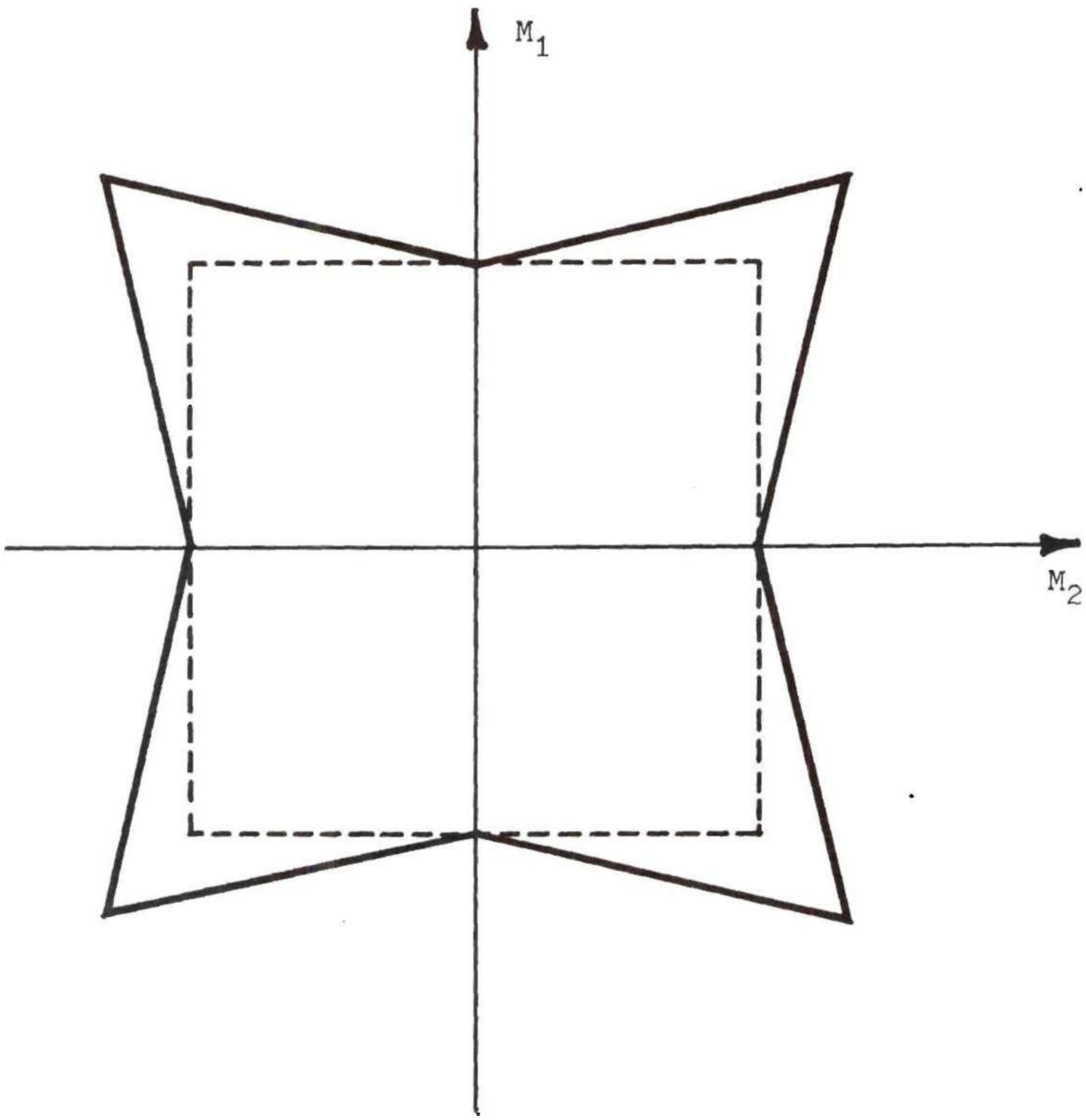


Figure 5. General Form of Baus and Tolaccia's Yield Criterion.

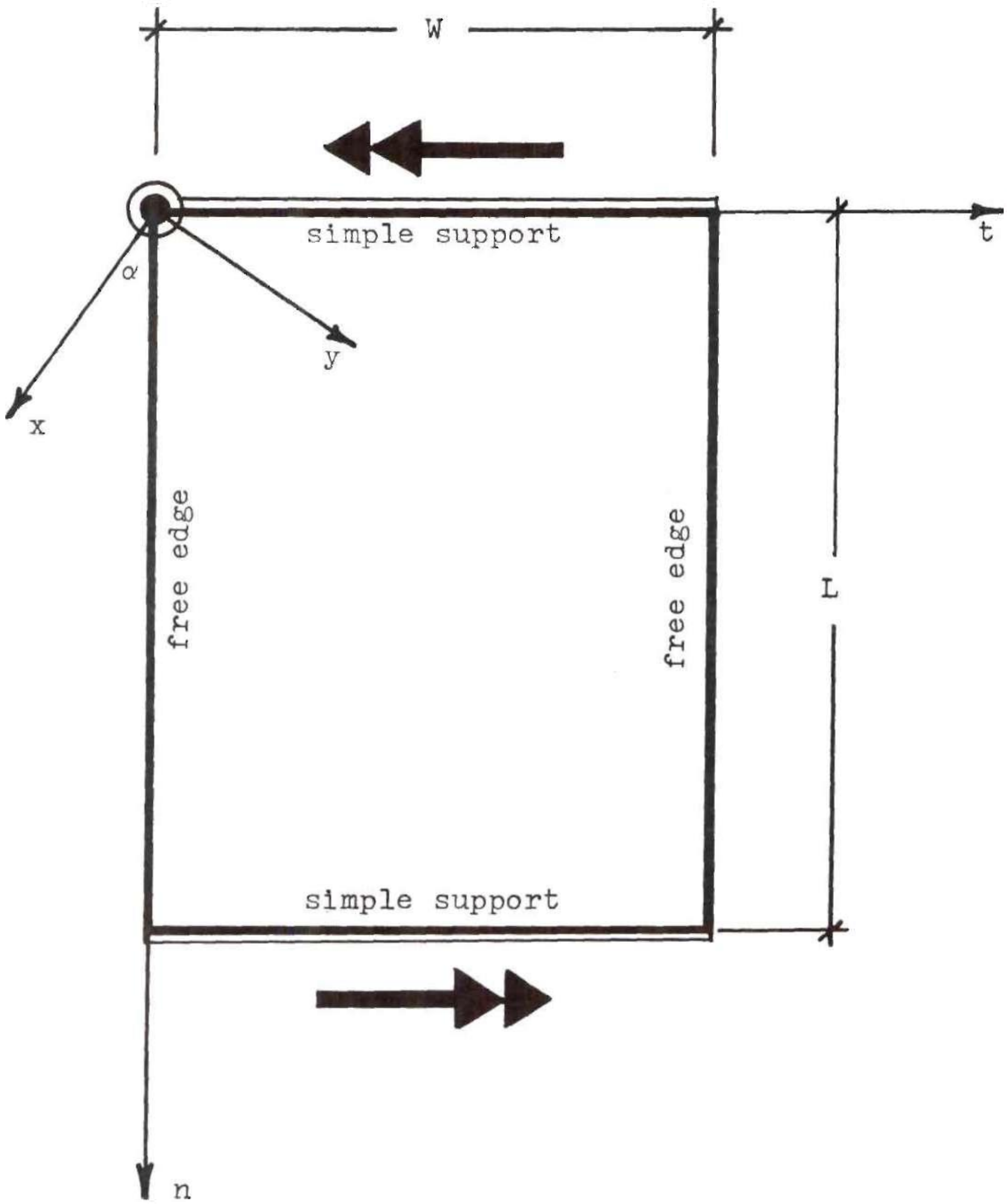


Figure 6. Plan View of Set-Up, Equivalent to Lenkei's, for an Orthotropic Slab.

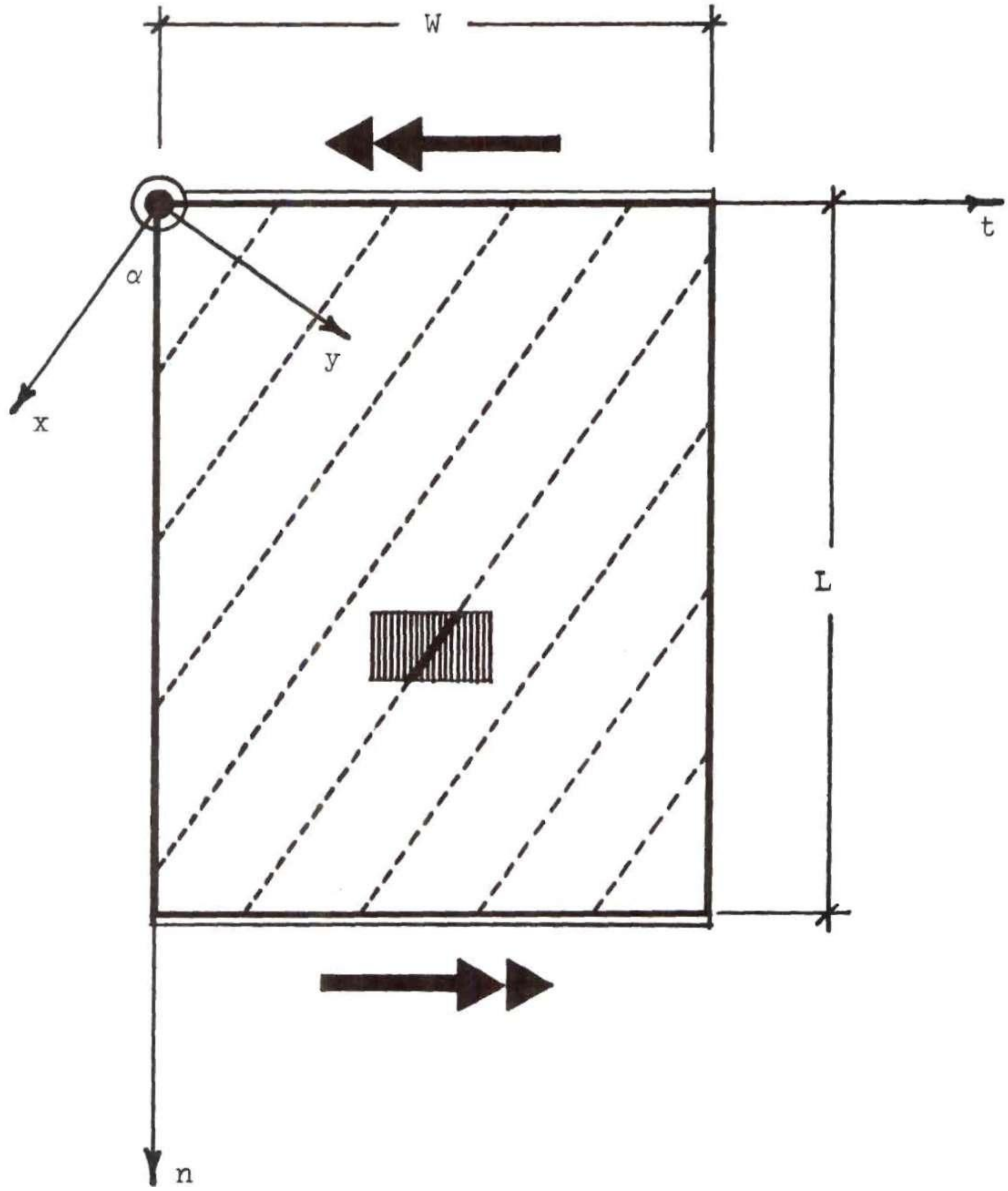


Figure 7. Plan View of Set-Up, Equivalent to Lenkei's for an Orthotropically Reinforced Concrete Slab.

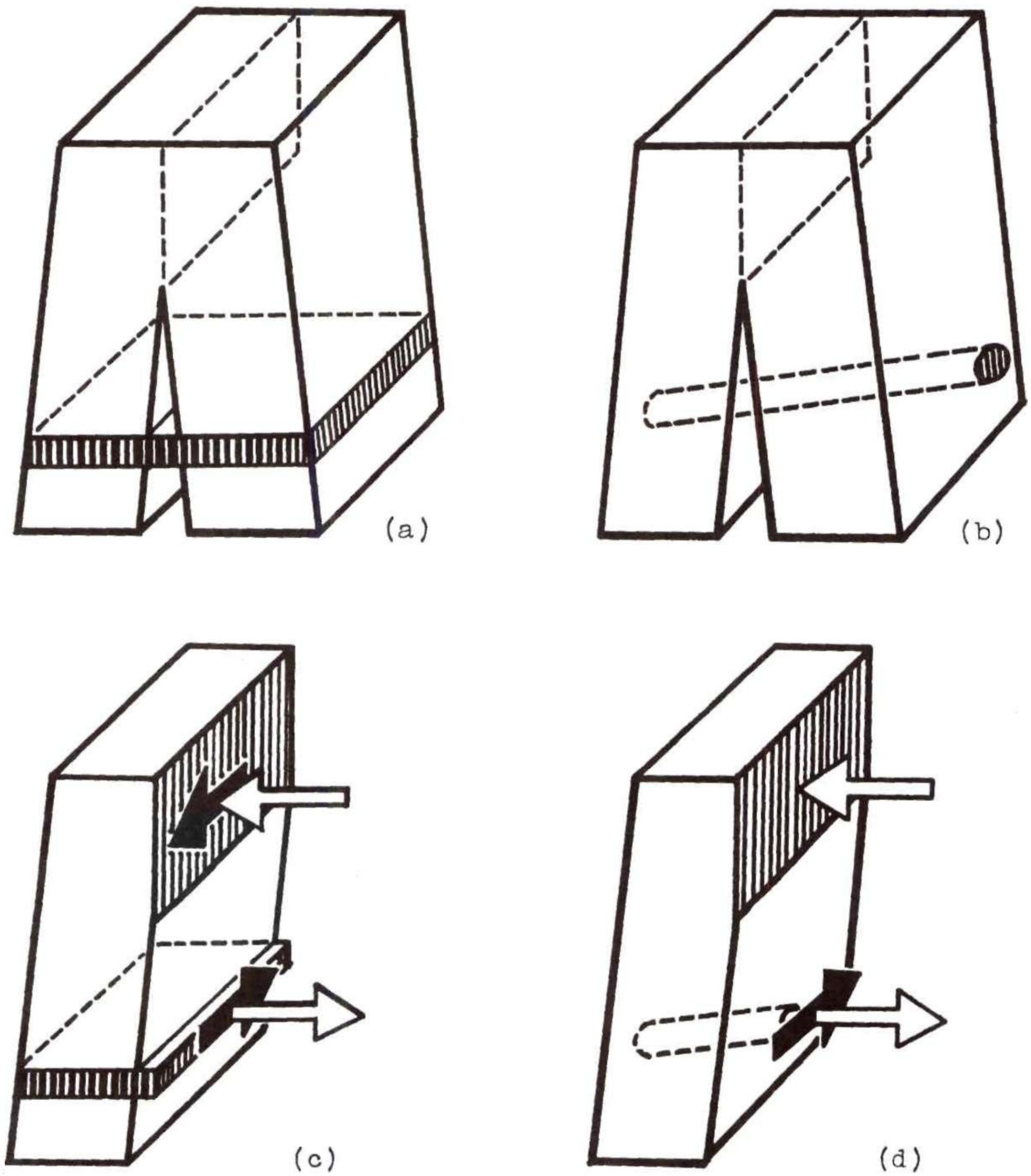


Figure 8. Typical Segment of Orthotropic Slab and of Orthotropically Reinforced Concrete Slab Showing Force Systems.

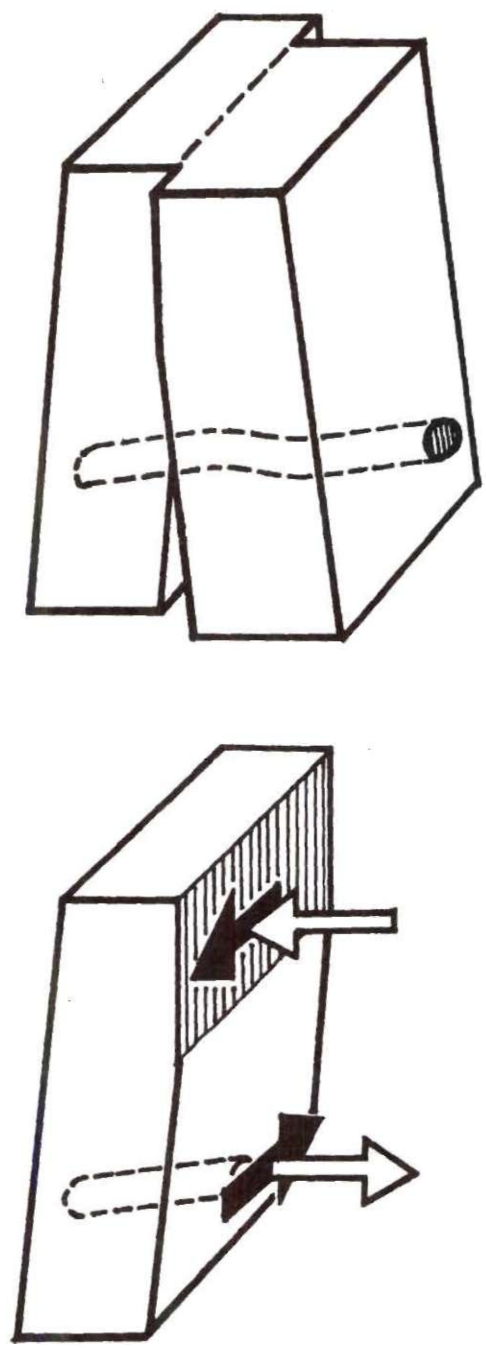


Figure 9. Typical Segment of Orthotropically Reinforced Concrete Slab Showing Effect of Modifying Kirchoff's Assumptions.

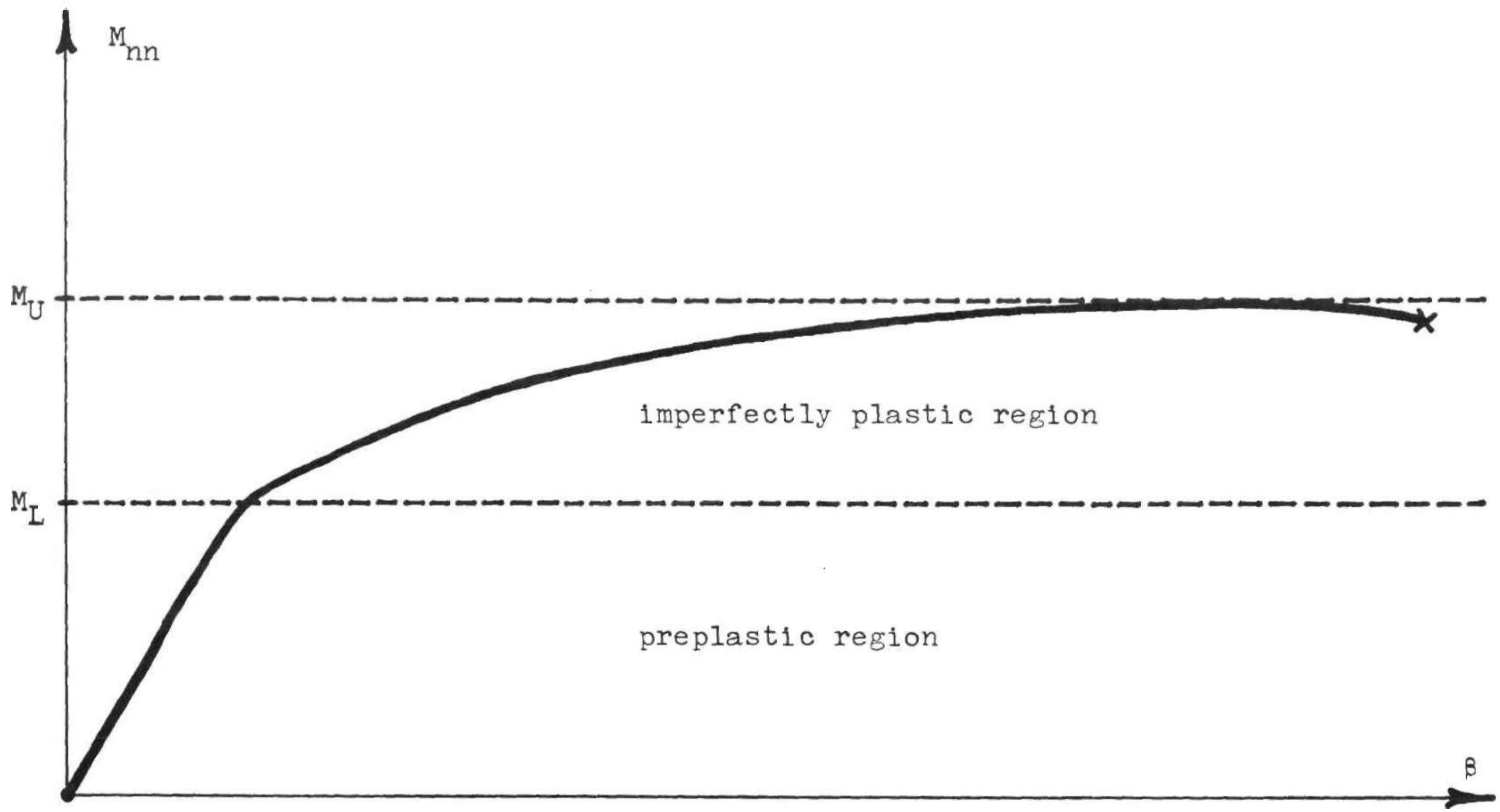


Figure 10. Typical Moment-Curvature Relationship for a Reinforced Concrete Slab with Reinforcement Normal to the Crack-Line.

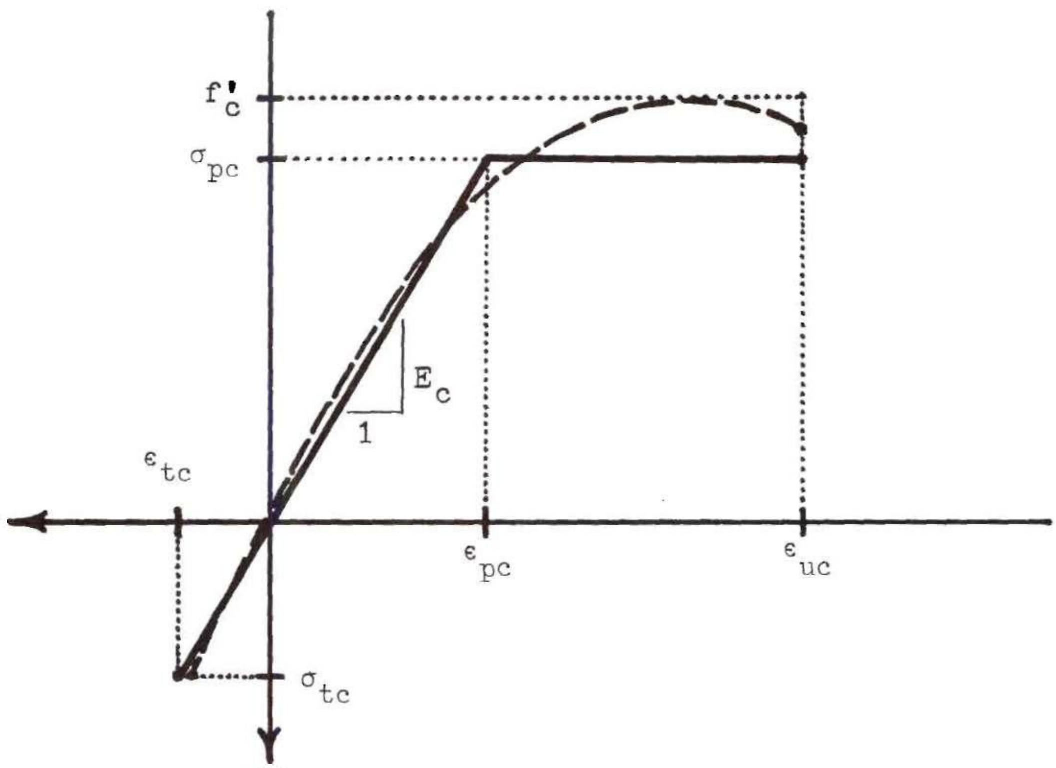
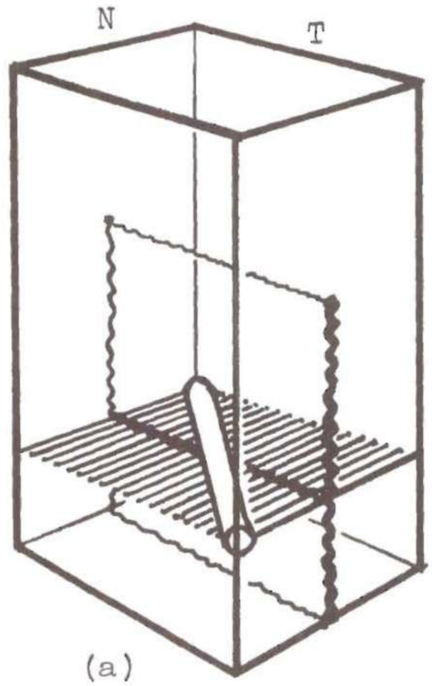


Figure 11. Typical and Idealized Stress-Strain Curve for Plain Concrete.



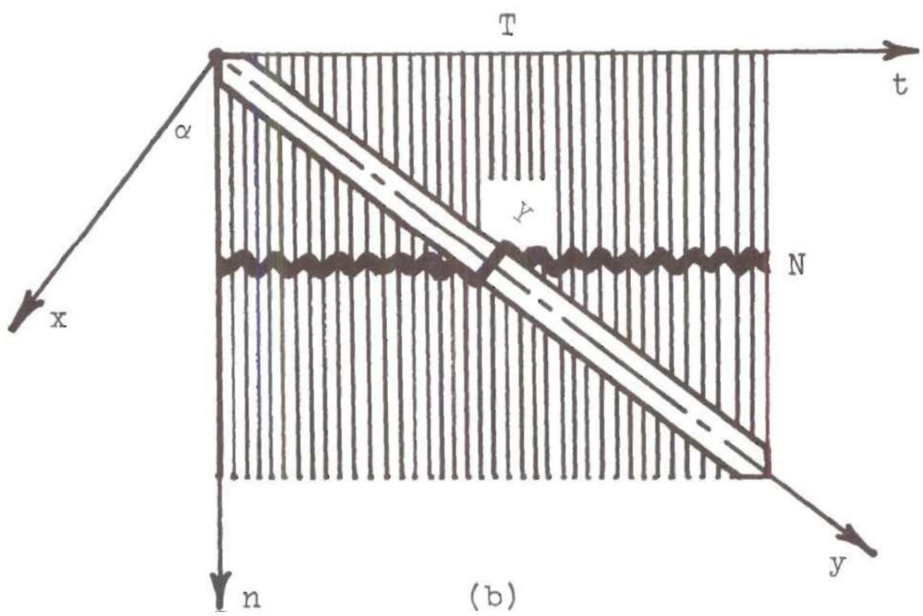


Figure 12. Depiction of Physical Relationship between the Reinforcing Bar, the Crack-Line and the Typical Segment.

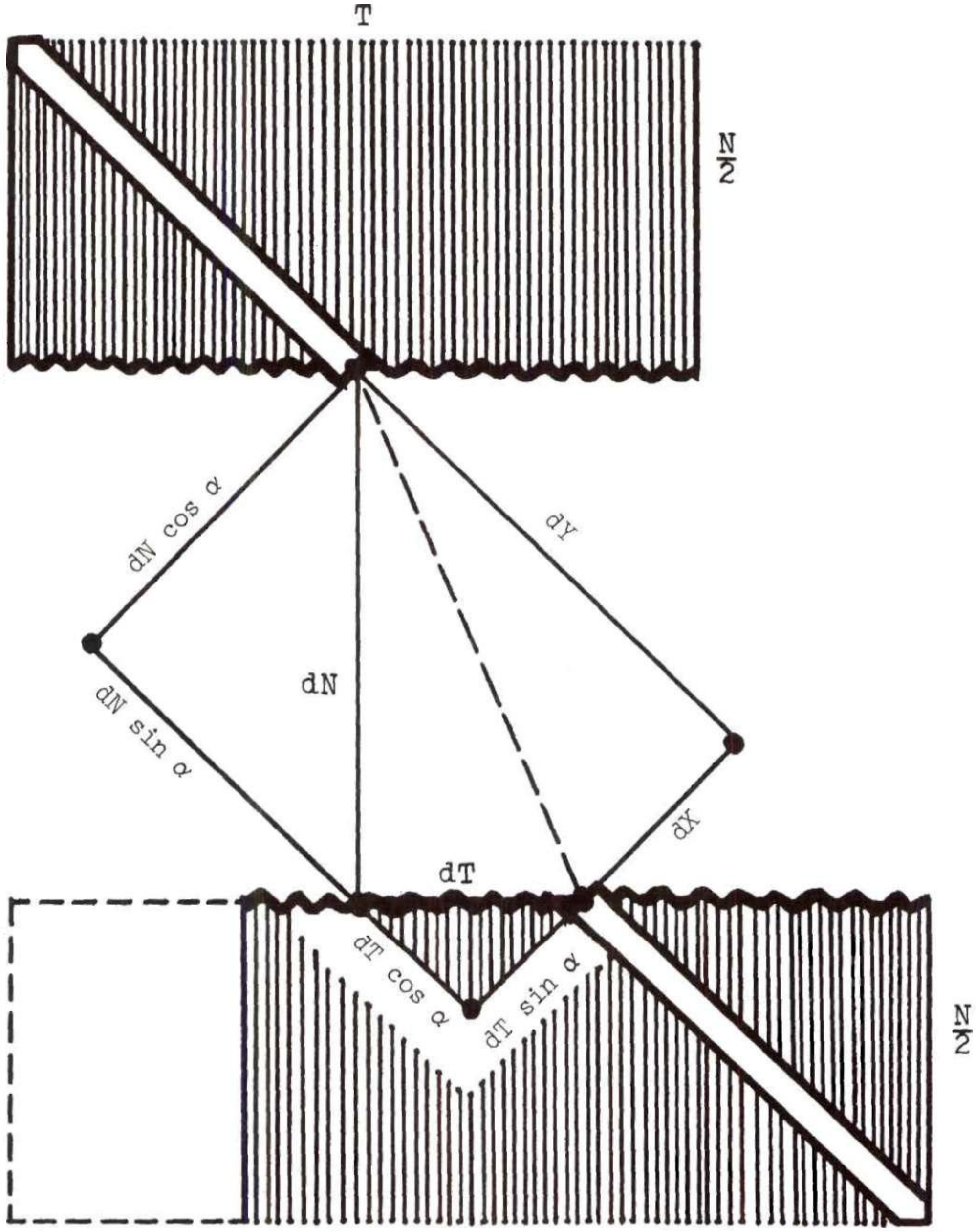


Figure 13. Geometry of Deformation for Reinforcing Bar Crossing an Opening Crack.

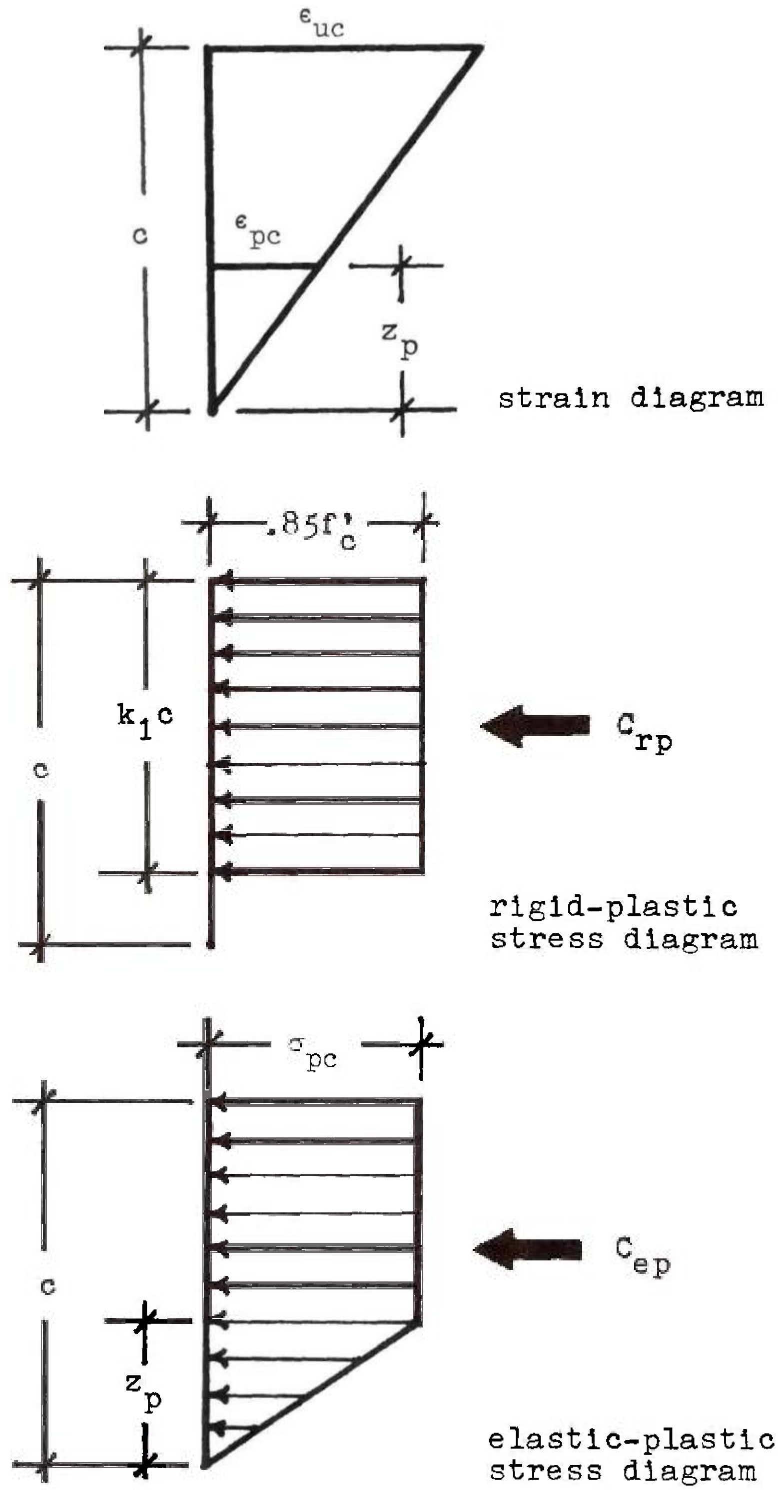


Figure 14. Idealized Normal Strain and Stress Distributions on Uncracked Concrete at Ultimate Moment.

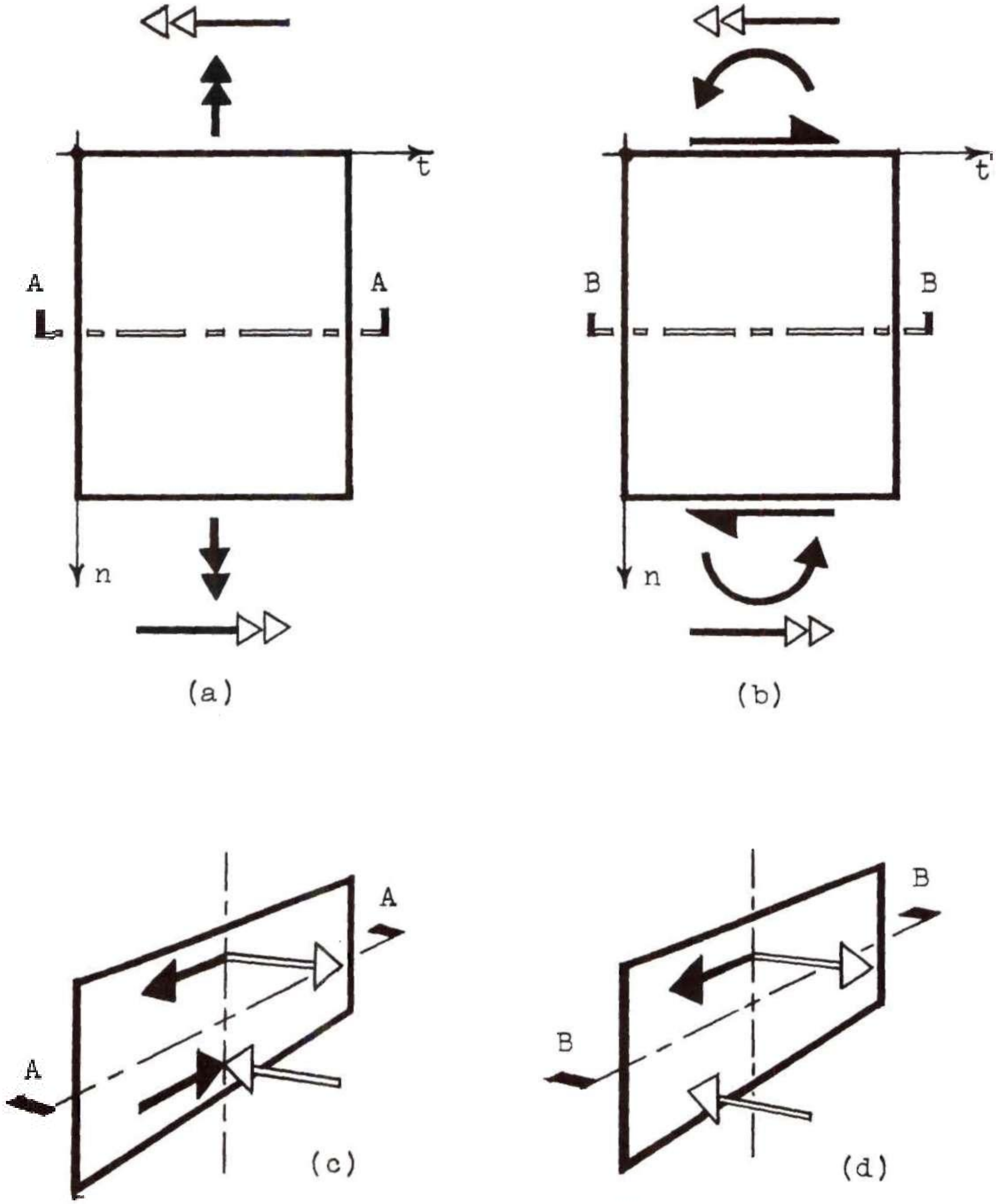


Figure 15. Force Distributions in Slabs with Different Boundary Conditions.

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